

TEXTBOOK SERIES

VOLUME 3

SYSTEMATIC OF RESERVOIR FLOW EQUATIONS

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actualized

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For kind Attention

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Chapter 1

General Formulation of the Fluid Flow Equation

1.1. Introduction

Reservoir engineering requires deep understanding in geology, rock and fluid mechanics, in fluid thermodynamics and a little knowledge and limited skills in mathematics. All practical underground fluid flow models are based on the same mathematical formulation, differently simplified to given problems. A reservoir engineer will never face difficulties, once he/she has understood the multicomponent, multiphase and multidimensional formulation of the flow equations and if he/she is able to adequately relate analytical solutions or numerical solution methods to his/her actual problems. A reservoir engineer should not spend efforts to learn the solution methods (e.g.: Laplace transform); all necessary solutions are known for more than a half century. Petroleum engineering students with limited affinity to mathematics but good practical mind often orient themselves to drilling and production technology, which is certainly against the industrial interest. The ability of a sound cognition of practical problems and their solution is more important in reservoir engineering than in other areas of petroleum technology.

The objective of this script is to give a short but comprehensive presentation of the reservoir flow equations. It will be shown that all established differential equations of the area have the same root and all of them can be derived easily from a most general formulation. These equations handle the problems of the two phase linear displacement, calculation of water influx, well test evaluation up to compositional numerical modeling (simulation). Investing once time and effort in this Chapter the petroleum engineer will be equipped for his/her entire professional life.

In this Section we deal with isothermal flow models only, therefore no enthalpy balance will be written. For the sake of generality and later extension, the temperature will be handled as an independent variable. Also the dual continuum formulation is excluded here, but it will be handled in Volume 5 in detail.

1.1.1 The Constitutive Equations

In physics and engineering, a constitutive equation or constitutive relation is a relation between

two physical quantities that is specific to a material or substance, and approximates the response of that material to external stimuli, usually such as applied fields or forces. Some constitutive equations are simply phenomenological, others are derived from "first principles". A common approximate constitutive equation frequently is expressed as simple proportionality using a parameter taken to be a property of the material. Such properties are also the permeability of the porous material and the diffusion coefficient. The first is defined by the Darcy's law, the second by the Fick's law. Both laws describe the transport of matter in an almost identical way: Flux is proportional to a gradient, and the constant of proportionality is a characteristic of the material.

1.1.2 The Darcy Law

The permeability of the porous material is defined by the phenomenological Darcy-equation:

$$\vec{u} = -\frac{1}{\mu} \bar{k} \nabla \Phi, \quad (1.1)$$

where \vec{u} is the filtration velocity, μ the dynamic viscosity and Φ the potential. To account for the directional dependence of the material the scalar permeability is generalized to a tensor:

$$\bar{k} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \text{ where } k_{ij} = k_{ji}. \quad (1.2)$$

\bar{k} , in general, is a symmetric tensor. Some upscaling methods can produce a non-symmetrical permeability tensor, which would mean that the Darcy flow shows shear effects. For such an effect no physical evidence exists. An asymmetric permeability tensor is solely the product of the upscaling method.

Constitutive relations are often modified to account for non-linear behaviors or inhomogeneity in the material. In this sense, the Darcy equation is used for multiphase flow too:

$$\vec{u}_p = -\lambda_p \bar{k} \nabla \Phi_p, \quad (1.3)$$

where \vec{u}_p is the filtration velocity of the phase p ,

$$\lambda_p = \frac{k_{rp}}{\mu_p} \quad (1.4)$$

λ_p is the phase mobility and μ_p the dynamic viscosity. In multiphase flow the pore space is occupied by fluids, separated by phase surfaces. The proportion of one phase in the pore volume

is defined as the phase saturation S_p . The relative permeability depends on these saturations:

$$k_{rp} = k_{rp}(\bar{S}_p). \quad (1.5)$$

The driving force is the gradient in the potential field:

$$\nabla\Phi_p = \nabla p_p - \rho_p \bar{g}, \quad (1.6)$$

where ρ_p is the phase density.

Inserting Equ.1.6 into Equ.1.3 results in:

$$\dot{u}_p = -\lambda_p \bar{k}(\nabla p_p - \rho_p \bar{g}) \quad (1.7)$$

Darcy's Law is valid for laminar flow through the porous medium. In a fine grained medium the dimensions of interstices are small and thus flow is laminar. Coarse-grained media behave similarly. But in very coarse-grained media - also in naturally fractured media - the flow is of turbulent nature. Hence, Darcy's Law is not valid in such media.

1.1.3 The Fick's Law

The first Fick's law relates the diffusive flux to the concentration gradient under the assumption of steady state. It postulates that the flux goes from regions of high concentration to regions of low concentration, with the magnitude proportional to the concentration gradient:

$$\dot{J}_c = -D_c \nabla C_c, \quad (1.8)$$

where J is the diffusion flux per unit area per unit time [$\text{kmol.m}^{-2}.\text{s}^{-1}$], D is the diffusivity [$\text{m}^2.\text{s}^{-1}$] and C the concentration [kmol.m^{-3}]. For a porous medium under multiphase condition this yields:

$$\dot{J}_{pc} = -\phi S_p D_{pc} \nabla C_{pc}, \quad (1.9)$$

where ϕ is the porosity and S_p the phase saturation. In anisotropic media, the diffusion coefficient depends on the direction and D becomes a symmetric tensor. By changing the concentration C to the mole fraction:

$$C_{pc} = D_p x_{pc}, \quad (1.10)$$

where D_p is the mole density [kmol.m^{-3}] of the phase p , the diffusion flux for the component c in phase p becomes:

$$\dot{J}_{pc} = -\phi S_p \mathbf{D}_{pc} \nabla (D_p x_{pc}) = -\phi S_p \mathbf{D}_{pc} (D_p \nabla x_{pc} + x_{pc} \nabla D_p). \quad (1.11)$$

If $x_{pc} \ll 1$ then $\nabla D_p \approx 0$ therefore the following form is a practical approximation:

$$\dot{J}_{pc} = -\phi S_p D_p \mathbf{D}_{pc} \nabla x_{pc} \quad (1.12)$$

1.1.4 Constraints and Dependencies

The porous material has a porosity ϕ , which is a function of the pressure (and temperature). The fluid system is composed of C components. The state of the fluid depends on the pressure p , the temperature T and the overall composition z . These state variables define the number of phases, the distribution of the material between the phases, the density and other fluid properties of the phases.

The fluid system builds P phases separated by phase surfaces. The composition of the p^{th} phase is given by the mole fraction x_{pc} . The mole fractions of the phases obey the equilibrium constrains:

$$x_{p'c} = K_{p'pc} x_{pc}, \quad (1.13)$$

where x_{pc} and $x_{p'c}$ are the mole fractions of the component c in phase p and p' , respectively. The equilibrium constant K is specific for the component c and the phases p and p' . K is dimensionless and depends on the pressure, temperature and the overall composition:

$$K_{p'pc} = K_{p'pc}(p_p, T, \bar{z}_c). \quad (1.14)$$

Naturally for all phases the ‘‘mole constraint’’ is valid:

$$\sum_{c=1}^C x_{pc} = 1; \quad p \in \{1, \dots, P\}. \quad (1.15)$$

The proportion of one phase in the pore volume is defined as the phase saturation S_p . It is obvious that

$$\sum_{p=1}^P S_p = 1; \quad (1.16)$$

Equ.1.16 is called ‘‘saturation constraint’’.

The specific mole density D_p [kmol.m⁻³] is

$$D_p = D_p(p_p, T, \bar{x}_{pc}), \quad (1.17)$$

where x_{pc} is the mole fraction of phase p . Let M_c be the mole weight of component c , then the phase density, ρ_p [kg.m⁻³] can be calculated easily:

$$\rho_p = D_p \sum_{c=1}^C M_c x_{pc}. \quad (1.18)$$

The difference between the phase pressures is the capillary pressure P_c [bar]:

$$p_p - p_{p'} = P_{cpp'} \quad p \neq p' \quad p, p' \in \{1, \dots, P\}. \quad (1.19)$$

The capillary pressure depends on the saturation and on the interfacial tension, which is a function of pressure, temperature and composition. If the interfacial tension is invariable, then the capillary pressure reduces to a function of the saturation only:

$$P_{cpp'} = P_{cpp'}(S_p) \quad p \neq p'. \quad (1.20)$$

Note, that the subscript c in the symbol of capillary pressure P_c has nothing in common with the usual component sign c .

1.1.5 The Equation of Continuity

Consider a cube with a side length of one meter. The mass of component c changes within this unit bulk volume with the following rate:

$$\delta G_c = \frac{\partial}{\partial t} \left(\phi \sum_{p=1}^P S_p D_p x_{pc} M_c \right). \quad (1.21)$$

The origin of the expression of Equ.1.21 is explained in Fig.1.1 The mass flow rate stimulated by convection changes over the cube:

$$\delta U_c = \nabla \left(\sum_{p=1}^P \hat{u}_p D_p x_{pc} M_c \right) \quad (1.22)$$

Equ.1.22 is explained in Fig.1.2. Based on the principle of mass conservation the following equivalency

$$-\delta U_c = \delta G_c \tag{1.23}$$

has to be satisfied, therefore:

$$-\nabla \left(\sum_{p=1}^P \dot{u}_p D_p x_{pc} M_c \right) = \frac{\partial}{\partial t} \left(\phi \sum_{p=1}^P S_p D_p x_{pc} M_c \right). \tag{1.24}$$

By considering mass transport by diffusion and the possibility of a volume source/sink the final form of Equ.1.24 can be written as:

$$-\nabla \left(\sum_{p=1}^P \dot{u}_p D_p x_{pc} M_c \right) - \nabla \left(\sum_{p=1}^P \dot{j}_{pc} M_c \right) + q_c M_c = \frac{\partial}{\partial t} \left(\phi \sum_{p=1}^P S_p D_p x_{pc} M_c \right) \tag{1.25}$$

Equ.1.24 and Equ.1.25 are displayed in Fig.1.3.

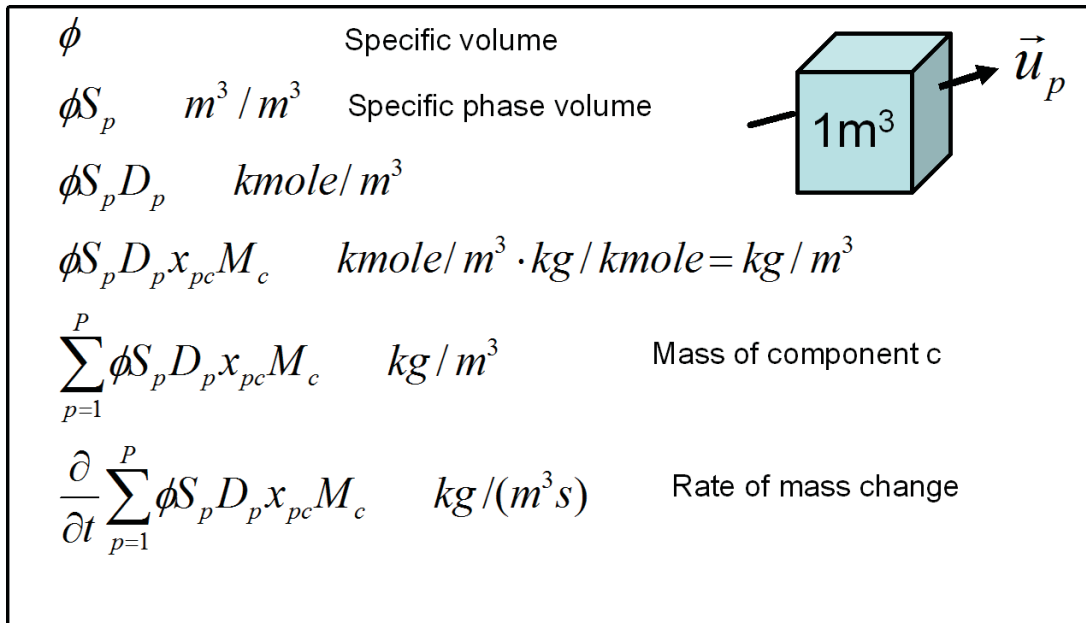
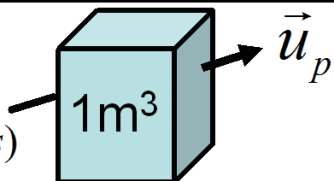
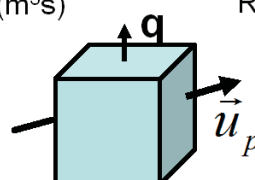


Figure 1.1: Explanation of accumulation term



$\vec{u}_p \quad m/s = m^3/(m^2s)$
 $\vec{u}_p D_p \quad m^3/(m^2s) \cdot kmole/m^3 = kmole/(m^2s)$
 $\vec{u}_p D_p x_{pc} M_c \quad kmole/(m^2s) \cdot kg/kmole = kg/(m^2s)$
 $\sum_{p=1}^P (\vec{u}_p D_p x_{pc} M_c) \quad kg/(m^2s) \quad \text{Mass velocity}$
 $\nabla \left[\sum_{p=1}^P (\vec{u}_p D_p x_{pc} M_c) \right] \quad kg/(m^3s) \quad \text{Rate of mass velocity change}$

Figure 1.2: Explanation of flow term



Change in mass flux $kg/(m^3s)$ Rate of mass change $kg/(m^3s)$
 $\nabla \left[\sum_{p=1}^P (\vec{u}_p D_p x_{pc} M_c) \right] \quad \frac{\partial}{\partial t} \sum_{p=1}^P \phi S_p D_p x_{pc} M_c$
 $-\nabla \left[\sum_{p=1}^P (\vec{u}_p D_p x_{pc} M_c) \right] = \frac{\partial}{\partial t} \sum_{p=1}^P \phi S_p D_p x_{pc} M_c$
 With Diffusion and volume source rate
 $-\nabla \left[\sum_{p=1}^P (\vec{u}_p D_p x_{pc} M_c) \right] - \nabla \left(\sum_{p=1}^P \vec{J}_{pc} M_c \right) + q_c M_c = \frac{\partial}{\partial t} \sum_{p=1}^P \phi S_p D_p x_{pc} M_c$

Figure 1.3: Explanation to the balance equation

The first term on the left-hand side in Equ.1.25 is the *convection term* or *flow term*. It gives the

mass of component c flowing through the unit surface per unit time (mass velocity). The summation over all phases is necessary, because the particular component may be present in several phases. The second term is the *diffusion term*. It gives the mass of the component c moving through the unit surface per unit time, driven by the concentration gradient. The third term is the *volumetric source/sink term*. It determines how much of component c will be taken out, or is supplied to the unit volume per unit time. The right-hand side is the *accumulation term*. It contains $S_p D_p x_{pc}$ moles and $S_p D_p x_{pc} M_c$ mass of component c . The summation over all phases gives the amount of component c within an unit bulk volume. J_{pc} [$\text{kmol.m}^{-2}.\text{s}^{-1}$] is the molar flux of component c in phase p , which can be expressed by Fick's first law, as given in Equ.1.11.

Equ.1.25 is a mass balance, which can be shortened by the constant mole weight M_c receiving a mole balance:

$$-\nabla \left(\sum_{p=1}^P \bar{u}_p D_p x_{pc} \right) - \nabla \left(\sum_{p=1}^P \dot{J}_{pc} \right) + q_c = \frac{\partial}{\partial t} \left(\phi \sum_{p=1}^P S_p D_p x_{pc} \right) \quad (1.26)$$

Thus, the mole and mass balance are equivalent. Inserting Darcy's Equ.1.7 and Fick's Equ.1.12 gives:

$$\nabla \left(\sum_{p=1}^P \lambda_p \bar{k} (\nabla p_p - \rho_p \bar{g}) D_p x_{pc} \right) + \nabla \left(\sum_{p=1}^P \phi S_p D_p D_{pc} \nabla x_{pc} \right) + q_c = \frac{\partial}{\partial t} \left(\phi \sum_{p=1}^P S_p D_p x_{pc} \right) \quad (1.27)$$

1.2. Initial and Boundary Conditions

The differential equations describe the changes of function values in space and in time. To get their values at a given time and at a given location an initial condition and boundary conditions have to be defined.

The initial condition defines the function values at any location within the domain Ω for time zero:

$$\Phi(\hat{x}, t = 0) = \Phi_0(\hat{x}), \quad \hat{x} \in \Omega. \quad (1.28)$$

Boundaries can be divided into inner and outer boundaries. Outer boundaries are the top and the bottom of the domain as outer delimitation lines (surfaces). Examples for inner boundaries are discontinuity lines (surfaces) such as faults.

Generally, two types of boundary conditions (BC) can be defined:

I. DIRICHLET type BC. - The potential is known at the boundary:

$$\Phi_{\Gamma}(t) = \Phi, \quad \hat{x} \in \Gamma. \quad (1.29)$$

II. VON NEUMANN type BC. - The flow rate across the boundary is known:

$$u_{p\Gamma}(t) = -\lambda_p \bar{k} \nabla \Phi_p \cdot \hat{n}_{\Gamma}, \quad \hat{x} \in \Gamma. \quad (1.30)$$

If the boundary is sealing (no-flow boundary) then:

$$u_{p\Gamma} = 0. \quad (1.31)$$

The top and bottom of the domain are sealing (no flow) boundaries.

At the outer delimitations of the domain all kinds of boundary conditions can be applied:

- The outer boundary is sealing.
- The flow rate as function of time is defined if an analytical aquifer model is used.
- Constant potential can be set for special cases.

At well surfaces both BCs can be set. The flow rate or the potential at such boundaries must be determined by a well model which considers the type of well operation (constant rate, constant bottom hole or well head pressure, etc.) and the pressure drop within the wellbore. Such a well model can be analytical, numerical or semi-analytical.

An inner boundary separates and connects two continua. At any point i of this boundary the flux must be continuous:

$$\bar{k}_{ir} \nabla \Phi_{pir} \cdot \hat{n}_i = \bar{k}_{il} \nabla \Phi_{pil} \cdot \hat{n}_i, \quad (1.32)$$

where ir and il denote the right and left hand sides of the discontinuity surface, Φ_p is the potential of phase p and \hat{n}_i is the unit normal vector to the surface. The flow through the inner boundary could cause a supplementary pressure drop (choke effect), therefore, the potential is not continuous at this boundary:

$$\Phi_{pir} = \Phi_{pil} - \varepsilon_i \hat{u}_{pi} \cdot \hat{n}_i, \quad (1.33)$$

where ε is rather a function of the potential difference $\Phi_{pir} - \Phi_{pil}$ than a constant. The filtration velocity of phase p can be calculated from both sides of the boundary:

$$\hat{u}_{pi} = -\lambda_p \bar{k}_{ir} \nabla \Phi_{pir}. \quad (1.34)$$

Chapter 2

Special Forms of the Balance Equation

We will show that all known equations of single and multiphase flow can be easily derived from Equ.1.25.

2.1. Multiphase Flow of Immiscible Fluids

Equ.1.24 gets, by inserting Equ.1.18, the following form:

$$-\nabla \cdot \left(\sum_{p=1}^P \frac{\vec{u}_p \rho_p x_{pc} M_c}{\sum_{c=1}^C M_c x_{pc}} \right) = \frac{\partial}{\partial t} \left(\phi \sum_{p=1}^P \frac{S_p \rho_p x_{pc} M_c}{\sum_{c=1}^C M_c x_{pc}} \right). \quad (2.1)$$

Let us consider that the composition of the phases remains unchanged. This is the case if a multicomponent oil phase will be displaced by brine (composed of water and salt). Both phases can be considered as a one component system. Oil and brine are so called pseudo-components, existing in one of the phases only. Consequently $x_{pc} = 1.0$,

$$\sum_{c=1}^C M_c x_{pc} = const, \quad (2.2)$$

and the summation over the phases is no longer necessary. Equ.2.1 gets the following form:

$$-\nabla \cdot (\vec{u}_p \rho_p) = \frac{\partial}{\partial t} (\phi S_p \rho_p). \quad (2.3)$$

2.2. Single phase flow

In this case $S_p = 1.0$, therefore

$$-\nabla(\vec{u}\rho) = \frac{\partial}{\partial t}(\phi\rho). \quad (2.4)$$

In the case of one-phase flow the relative permeability $k_{rp} = 1$. Inserting Equ.1.7 into Equ.2.4 we get

$$\nabla\left(\frac{\bar{k}\rho}{\mu}\nabla p\right) - \nabla\left(\frac{\bar{k}\rho^2}{\mu}\vec{g}\right) = \frac{\partial}{\partial t}(\phi\rho). \quad (2.5)$$

The second term on the left hand side is small compared to the first one and usually it is neglected:

$$\nabla\left(\frac{\bar{k}\rho}{\mu}\nabla p\right) = \frac{\partial}{\partial t}(\phi\rho). \quad (2.6)$$

2.2.1 Single phase low compressible fluid flow

The isothermal compressibility of a homogenous fluid is

$$c = \frac{1}{\rho} \frac{d\rho}{dp}. \quad (2.7)$$

The fluid is categorized as low compressible if the compressibility factor c [$\text{m}^3 \cdot \text{bar}^{-1}$] is small and constant. After integration of Equ.2.7, yields:

$$\rho = \rho_o e^{c(p-p_o)}, \quad (2.8)$$

where: ρ_o is the density of fluid at any reference pressure (p_o). Applying TAYLOR's rule and neglecting the terms of higher order we get the following approximation:

$$\rho \approx \rho_o \cdot [1 + c(p-p_o)]. \quad (2.9)$$

Based on Equ.2.8 the following transformation can be made:

$$\rho \nabla p = \rho_o e^{c(p-p_o)} \nabla p = \frac{1}{c} \nabla \left(\rho_o e^{c(p-p_o)} \right) = \frac{1}{c} \nabla \rho. \quad (2.10)$$

Substituting Equ.2.10 into Equ.2.6 becomes:

$$\nabla \left[\frac{\bar{k}}{c\mu} \nabla \rho \right] = \frac{\partial}{\partial t} (\phi \rho). \quad (2.11)$$

Differentiation of Equ.2.9 gives:

$$\nabla \rho = \rho_o c \nabla p. \quad (2.12)$$

Substituting both Equ.2.12 and Equ.2.9 into Equ.2.11 results in:

$$\nabla \left[\frac{\bar{k}}{\mu} \rho_o \nabla p \right] = \frac{\partial}{\partial t} \{ \phi \rho_o [1 + c(p - p_o)] \}. \quad (2.13)$$

In Equ.2.13 ρ_o , c and p_o are constant.

2.2.2 Single phase low compressible fluid flow in low compressible porous media

Similar approximation as to the fluid can be applied for the porosity too:

$$\phi = \phi_o (1 + c_\phi (p - p_o)). \quad (2.14)$$

Now

$$\phi \rho = \phi_o \rho_o (1 + c(p - p_o))(1 + c_\phi (p - p_o)) \approx \phi_o \rho_o (1 + c_t (p - p_o)) \quad (2.15)$$

where

$$c_t = c + c_\phi \quad (2.16)$$

is the total compressibility. The cross product $c c_\phi$ is with 5 magnitudes less than c_t therefore the corresponding term was neglected. The viscosity can be regarded as constant as well ($\mu = const$), hence the fluid compressibility is small.

The substitution of Equ.2.12 into Equ.2.11 results in:

$$\nabla [\bar{k} \nabla p] = \frac{\phi_o c_t \partial p}{\mu \partial t}. \quad (2.17)$$

2.2.3 Single phase low compressible fluid flow in low

compressible isotropic and homogenous porous media

In a isotropic and homogenous porous media the permeability k is a constant scalar value. Therefore the Equ.2.17 becomes:

$$\nabla^2 p = \Delta p = \frac{1}{K} \frac{\partial p}{\partial t}, \quad (2.18)$$

where

$$K = \frac{k}{\mu c_t \phi_0} \quad (2.19)$$

is the piezometric conductivity and

$$c_t = c + c_\phi. \quad (2.20)$$

2.2.3.1 Low compressible fluid flow in radial symmetrical coordinate system

In one dimensional radial symmetrical coordinate system

$$\nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad (2.21)$$

and so Equ.2.18 get the following form:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{1}{K} \frac{\partial p}{\partial t}. \quad (2.22)$$

2.2.4 Incompressible or steady state flow

In this case the total compressibility c_t is zero, K equals to ∞ and the Equ.2.18 reduces to

$$\nabla^2 p = \Delta p = 0. \quad (2.23)$$

Incompressible linear flow:

$$\frac{\partial^2 p}{\partial x^2} = 0. \quad (2.24)$$

2.3. Two-Phase Two-Component Incompressible Flow

2.3.1 The Pressure and Saturation Equations

We assume that the two components are immiscible, forming two incompressible phases. Let phase 1 be the displacing water and phase 2 the displaced oil. Because ρ_w, ρ_o and ϕ , are constant values, the balance equations according Equ.2.3 are:

$$-\nabla \vec{u}_w = \phi \frac{\partial S_w}{\partial t} \quad \text{for water,} \quad (2.25)$$

$$-\nabla \vec{u}_o = \phi \frac{\partial S_o}{\partial t} \quad \text{for oil.} \quad (2.26)$$

Inserting Equ.1.7:

$$\nabla \left\{ \lambda_w \bar{k} (\nabla p_w - \rho_w \vec{g}) \right\} = \phi \frac{\partial S_w}{\partial t}, \quad (2.27)$$

$$\nabla \left\{ \lambda_o \bar{k} (\nabla p_w + \nabla P_{cow} - \rho_o \vec{g}) \right\} = \phi \frac{\partial S_o}{\partial t}. \quad (2.28)$$

Neglecting the gravitation terms (i.e.: assuming two-dimensional horizontal flow) we get from Equ.2.27 the saturation equation, which is represented by Equ.2.29. Summing up Equ.2.27 and Equ.2.28 yields the pressure equation, shown by Equ.2.30:

$$\phi \frac{dS_w}{dt} - \bar{k} \nabla (\lambda_w \nabla p_w) = 0, \quad (2.29)$$

$$\nabla \bar{k} (\lambda_t \nabla p_w + \lambda_o \nabla P_c) = 0, \quad (2.30)$$

where $\lambda_t = \lambda_w + \lambda_o$ is the total mobility.

2.3.2 Fractional Flow Equation

The overall filtration velocity is

$$\vec{u} = \vec{u}_w + \vec{u}_o. \quad (2.31)$$

The incompressibility of the phases

$$\nabla \vec{u} = 0 \quad (2.32)$$

is valid, which means that \vec{u} is independent of the space coordinates x, y, z . Regarding one dimensional flow, the fractional value can be defined:

$$f_w = \frac{u_w}{u}. \quad (2.33)$$

Inserting Equ.2.33 into Equ.2.25 we get the following form:

$$\frac{\partial}{\partial x}(f_w u) + \phi \frac{\partial S_w}{\partial t} = 0. \quad (2.34)$$

Remember that u is independent of x , therefore:

$$\frac{u}{\phi} \frac{\partial f_w}{\partial x} + \frac{\partial S_w}{\partial t} = 0. \quad (2.35)$$

Neglecting the capillary pressure, f_w depends on the saturation only:

$$f_w = f_w(S_w), \quad (2.36)$$

and so the final form of Equ.2.35 is the well known BUCKLEY-LEVERETT equation:

$$\frac{u}{\phi} \frac{df_w}{dS_w} \frac{\partial S_w}{\partial x} + \frac{\partial S_w}{\partial t} = 0. \quad (2.37)$$

With appropriate boundary and initial conditions Equ.2.23, Equ.2.22 and Equ.2.37 can be solved analytically. These solutions are widely used for classical reservoir problems, evaluating well tests, calculating displacement efficiency, etc. The general flow Equ.1.27 is strongly non-linear and can be solved using numerical methods only. The solution will be calculated for discrete points in space and time.

Chapter 3

Black Oil Model in Compositional Formulation

Using conventional black oil type reservoir fluid description one considers 3 components (water, oil and gas) forming three phases: water, oil and gas. The indices

$$p = w, o, g; \quad P = 3$$

and

$$c = w, o, g; \quad C = 3$$

will be used to identify the phases and the components. The water and gas phases are formed by one component; the oil phase consists of two components: oil and gas. Based on Equ.1.16:

$$S_w + S_o + S_g = 1, \quad (3.1)$$

And from Equ.1.15 follows:

$$x_{ww} = 1, \quad (3.2)$$

$$x_{gg} = 1, \quad (3.3)$$

$$x_{oo} + x_{og} = 1. \quad (3.4)$$

In Equ.3.2 to Equ.3.4 the first index refers to the phase, the second one to the component. The reservoir oil is composed of $1m^3$ stock tank oil and R_s standard m^3 gas. Therefore the mole fractions are:

$$x_{oo} = \frac{\frac{\rho_o^0}{M_o}}{\frac{\rho_o^0}{M_o} + \frac{R_s \rho_g^0}{M_g}}, \quad (3.5)$$

$$x_{og} = \frac{\frac{R_s \rho_g^0}{M_g}}{\frac{\rho_o^0}{M_o} + \frac{R_s \rho_g^0}{M_g}}. \quad (3.6)$$

Where ρ_m^0 ($m = w, o, g$) is the standard density of water, oil and gas, respectively. Dividing Equ.3.6 by Equ.3.5 the solution gas ratio can be expressed in terms of mole fractions:

$$R_s = \frac{x_{og} M_g \rho_o^0}{x_{oo} M_o \rho_g^0}. \quad (3.7)$$

The mole densities can be calculated from the formation volume factors:

$$D_w = \frac{\rho_w^0}{M_w B_w}, \quad (3.8)$$

$$D_g = \frac{\rho_g^0}{M_g B_g}, \quad (3.9)$$

$$D_o = \left(\frac{\rho_o^0}{M_o} + \frac{R_s \rho_g^0}{M_g} \right) \frac{1}{B_o}. \quad (3.10)$$

Inserting Equ.3.7 in Equ.3.10 yields

$$D_o = \frac{\rho_o^0}{M_o} \left(1 + \frac{x_{og}}{x_{oo}} \right) \frac{1}{B_o}, \quad (3.11)$$

and using Equ.3.4 results in:

$$D_o x_{oo} = \frac{\rho_o^0}{M_o B_o}. \quad (3.12)$$

Inserting Equ.3.2 and Equ.3.8 into Equ.1.27 we get the balance equation for the water component:

$$\nabla[\lambda_w^* \bar{k} \nabla \Phi_w] + q_w = \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right). \quad (3.13)$$

The summation over the phases is not necessary because no water exists in oil and gas phase. The balance equation for oil is obtained by inserting Equ.3.4 and Equ.3.11 into Equ.1.23:

$$\nabla[\lambda^*_o \bar{k} \nabla \Phi_o] + q_o = \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right). \quad (3.14)$$

For the gas balance equation the summation over the oil and gas phase is necessary and it is obtained by inserting Equ.3.3, Equ.3.4, Equ.3.9 and Equ.3.10 into Equ.1.27:

$$\nabla[\bar{k}(\lambda^*_g \nabla \Phi_g + \lambda^*_o R_s \nabla \Phi_o)] + q_g = \frac{\partial}{\partial t} \left(\frac{\phi S_g}{B_g} + \frac{\phi S_o R_s}{B_o} \right). \quad (3.15)$$

The phase mobility λ^* is defined by:

$$\lambda^*_p = \frac{k_{rp}}{\mu_p B_p} \quad (p = w, o, g), \quad (3.16)$$

where the sign * indicates the difference to the usual mobility definition according to Equ.1.4. The phase potentials are:

$$\Phi_w = p_w - \rho_w g z, \quad (3.17)$$

$$\Phi_o = p_o - \rho_o g z, \quad (3.18)$$

$$\Phi_g = p_g - \rho_g g z, \quad (3.19)$$

where z is the depth. The differences between the phase pressures are the capillary pressures:

$$p_o - p_w = P_{cow}, \quad (3.20)$$

$$p_g - p_o = P_{cgo}. \quad (3.21)$$

4 Nomenclature

Symbols

B_p	-	formation volume factor, [m^3/m^3]
D_p	-	specific mole density, [kmol/m^3]
$D_{pp'}$	-	diffusion coefficient, [m^2/d]
\vec{g}	-	gravity acceleration vector, [m/s^2]
f	-	fraction value, [-]
J	-	diffusion flux, [$\text{kmol}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$]
\bar{k}	-	permeability tensor
k_{rp}	-	relative permeability, [-]
K	-	piezometric conductivity, [$\text{m}^2\cdot\text{s}^{-1}$]
M_c	-	molar mass of component c , [kg/mol]
\vec{n}_i	-	is the unit normal vector to a surface
$P_{cpp'}$	-	capillary pressure between phase p and p' , [bar]
p_p	-	phase pressure, [bar]
q_c	-	source/sink term for component c , [kmol/day]
q_p	-	source/sink term for component c , [kmol/day]
q_{cmf}	-	matrix-fracture transfer term, [kmol/day]
R_s	-	solution gas GOR, [-]
S_p	-	phase saturation, [-]
t	-	time, [day]
T	-	temperature, [K]
\vec{u}_p	-	filtration velocity, [m/day]
V_I	-	volume of gridblock I , [m^3]
x_{pc}	-	mole fraction of component c in phase p , [-]
r	-	radial distance, [m]
z	-	depth, [m]

Greek Symbols

ϕ	-	porosity, [-]
Φ_p	-	phase potential, [bar]
λ_p	-	phase mobility, [$1/\text{cp}$]
μ_p	-	phase viscosity, [cp]
ρ_p	-	phase density, [kg/m^3]

ρ_s	-	rock density, [kg/m ³]
$\sigma_{pp'}$	-	surface tension between phase p and p'
τ_I^J	-	block transmissibility, [m ³]
τ_{IJ}	-	interblock transmissibility, [m ³]

Subscripts

c	-	component
g	-	gas
n	-	old time level
$n+1$	-	new time level
o	-	oil
p	-	phase
w	-	water

Superscripts

N_c	-	total number of components
N_p	-	total number of phases
v	-	old iteration level
$v+1$	-	new iteration level

Conversion Factors

bar	=	psia*0.06894757
m	=	ft*0.3048
kg	=	lb*0.453592
1000m ³	=	MMSCF*26.795
kg/m ³	=	lb/ft ³ *16.01846
°F	=	°R-459.67
°C	=	$\frac{(\text{°F}-32)}{1.8}$
°API	=	$\frac{141.5}{\text{Spec. Gravity}-131.5}$

5 References

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