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Abstract

It is claimed in almost every experimental work on ceramics that the strength is Weibull distributed. The literature demonstrates that this is not valid in any case, but it is up to now the backbone in the design of brittle components. An overview on situations that deviate from Weibulls statistics is presented (multi-modal flaw distribution, *R*-curves, etc.). It is also shown that testing specimens with different volumes may help to understand the real strength distribution.

Inaccuracies that arise from using the Weibull's theory are presented. Monte Carlo simulations on the basis of the standardised testing procedure (30 specimens) clearly reveal that these deviations can be hardly detected on the basis of small samples.
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Keywords: Brittle failure; Weibull statistics; Monte Carlo

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Fracture statistics of ceramics
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Abstract

Weibull statistics is up to now the backbone in the mechanical design procedure of ceramic components but is not generally valid. Different situations that deviate are presented, e.g. materials having a multimodal flaw distribution or having an increasing R - curve.

It is shown by means of Monte Carlo simulations that on the basis of the standardised testing procedure (30 specimens) is not possible to decide whether a material is Weibull distributed or not. Measuring the strength of specimens with different size allows –after reasonable experimental effort- the determination of the strength distribution in a reliable way and in a wide range of parameters.

Keywords

Brittle failure, Weibull statistics, Monte Carlo

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1. Introduction

Fracture of brittle materials (e.g. ceramics) usually initiates from flaws [1], which are distributed in the material. The strength of the specimen depends on the size of the major flaw, which varies from specimen to specimen. Therefore, the strength of brittle materials has to be described by a probability function (statistics) [2-5]. It follows from the experiments that the probability of failure increases with load amplitude and with size of the specimens [2, 3, 6]. The first observation is trivial. The second observation follows from the fact that it is more likely to find a major flaw in a large specimen than in a small one. Therefore the mean strength of a set of large specimens is smaller than the mean strength of a set of small specimens. This size effect on strength is the most prominent and relevant consequence of the statistical behaviour of strength in brittle materials.

For more than sixty years Weibull proposed a statistical theory of brittle fracture [7, 8]. His fundamental assumption was the weakest link hypothesis, i.e. the specimen fails, if its weakest volume element fails. Using some empirical arguments necessary to obtain a simple and good fitting to his experimental data, he derived the so-called Weibull distribution function, which - in its simplest form, for a uniaxial homogenous tensile stress state (stress amplitude σ) and for specimens of the volume, V, - is given by:

$$F(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad . \tag{1}$$

The Weibull modulus m is a measure for the scatter of strength data: the wider the distribution is the smaller is m. σ_0 is a characteristic strength value and V_0 the chosen normalising volume.

Up to these days the Weibull distribution function is the basis of the state of the art in mechanical design procedure of ceramic components. It should be noted that in almost every experimental study on fracture statistics of ceramics is claimed that the data are Weibull distributed[1]. But this is not necessarily true, because -for that type of statement- the sample size is too small in any case[2]. To show this is a major goal of this paper.

The strength testing of ceramics and the determination of the Weibull distribution are standardised [10, 11]. Following the standards the Weibull distribution function has to be measured on a sample of "at least" 30 specimens (due to high machining costs, larger samples

1 Eq. 1 defines the so called two-parameter form of the Weibull distribution. Weibull also proposed a three parameter form, where σ in Eq. 1 is replaced by the expression $(\sigma - \sigma_u)$. For stresses below a tensile threshold stress σ_u failure cannot occur. Although this threshold stress reflects the wishes of ceramic engineers for reliable components, the two parameter form of the Weibull distribution seems to reflect the observed behaviour of ceramic materials better as the three parameter form (see for example Lu et al [9]). For specimens with significant internal stress fields, the threshold stress may reflect the influence of these stresses. For a given set of data the two parameter equation gives an upper limit and makes therefore a "safe" reliability analysis possible.
2 The same statement is also valid for the three parameter form. Especially the fit of the strength stress - if it is based on a small sample - is in general not stable.

are hardly tested in the daily testing practice). The range of "measured" failure probabilities, F, increases with the sample size [12] and is - for a sample of 30 tests - very limited (between ~ 1/60 and ~ 59/60). To determine the design stress, the measured data have - in general - to be extrapolated to the "tolerated" failure probability of the components (~ $1/10^6$) and to the (effective) volume respectively, which often results in a very large extrapolation span [12].

In the last fifty years a significant amount of research has been made to give Weibull's empirical theory a more fundamental basis [13-21]. The paper of Kittl and Diaz [22] gives an excellent overview on the former developments. Freudenthal [15] showed for homogenous and brittle materials that, if the flaws do not interact (i.e. if they are sparsely distributed), the probability of failure only depends on the number of destructive flaws, $N_{c,S}$, occurring in a specimen of size and shape, S,

$$F_S(\sigma) = 1 - \exp\left[- N_{c,S}(\sigma)\right]. \tag{2}$$

In this equation, $N_{c,S}$ denotes the mean number of destructive (critical) flaws in a large sample (i.e. the value of expectation). Jayatilaka and Trustrum [13] demonstrated in their noteworthy paper, that, for a brittle and homogeneous material, the distribution of the strength data is caused by the distribution of sizes and orientations of the flaws, and that a Weibull distribution of strength will be observed for flaw populations with a monotonically decreasing density of flaw sizes. Danzer et al. [18-20] applied their ideas to flaw populations with any size distribution and to specimens with an inhomogeneous flaw population. Then, the strength distribution strongly depends on the shape of the flaw distributions in the material. Again, it was necessary to assume that a specimen fails if any flaw initiates fracture (the weakest link hypothesis), and if there is no interaction between flaws. On the basis of these ideas a direct correlation between the flaw size distribution and the scatter (statistics) of strength data can be recognised.

In this paper correlations between flaw size distribution and fracture statistics are discussed. It will be shown that, for many reasons, the fracture statistics may strictly deviate from the Weibull statistics, Eq. 1. It also will be shown by means of Monte Carlo simulations that these deviations can hardly be detected on the basis of small samples but that the testing of specimens of different size may offer a way out of that dilemma.

2. Flaw size distribution and strength distribution

For design purposes the explicit dependence of the number of destructive flaws and the applied load has to be known. If not differently mentioned a homogeneous tensile stress field (stress amplitude σ) is assumed in the following discussions. For simplicity and without loss of generality it is also assumed that the flaws can be described by a single parameter a which characterises their size, (e.g. the crack radius) and that the crack like flaws are perpendicularly oriented to the stress direction (an example how to account for orientation effects is given in the paper of Jayatilaka and Trustrum [13]). The relative frequency of flaws sizes (flaw size a) can be described by a function $g(a, \boldsymbol{r})$ [defects/m^4], which may –in general– also depend on the position vector $\boldsymbol{r}$. (see Fig. 1)

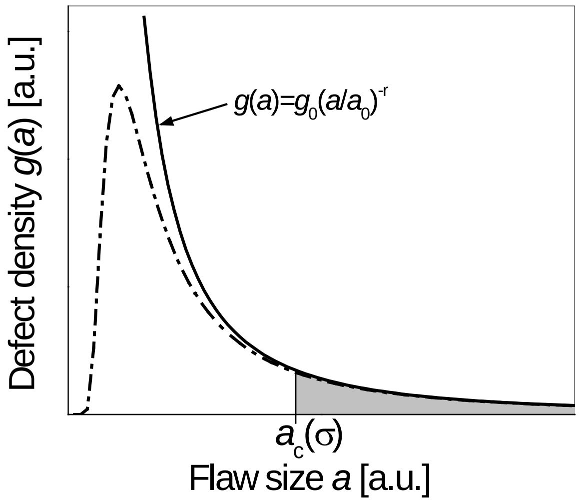


Fig. 1: Relative density of flaw sizes versus flaw size. The dashed line shows a typical distribution, the full line the behaviour necessary for a Weibull distribution. The dashed area gives the density of destructive flaws. The left bound shifts to the left if the stress increases.

A failure criterion connects the size of the flaw with a critical load. E.g. the Griffith/Irwin criterion predicts that crack like flaws gets critical, if their stress intensity factor $K = \sigma Y \sqrt{\pi a}$ exceeds the fracture toughness K_{Ic} [3, 4]:

$$K \geq K_{Ic} \quad , \tag{3}$$

Y is a geometric factor, which – for flaws which are small compared to the specimen size – is of the order of one. The critical crack size is then:

$$a_c = \frac{1}{\pi} \cdot \left(\frac{K_{Ic}}{Y \cdot \sigma} \right)^2 \quad . \tag{4}$$

Fig. 1 shows an example for a relative flaw size density function versus the flaw size (dashed line). All flaws larger than the critical value $a_c(\sigma)$ will cause failure. Their density is given by the integral (shaded area):

$$n_c(\sigma, \boldsymbol{r}) = \int_{a_c(\sigma)}^{\infty} g(a, \boldsymbol{r}) \, \mathrm{d}\, a \, . \tag{5}$$

It should be noted that the lower integration limit would shift to the left if the stress increases. Therefore, giving a higher density of destructive flaws at higher stresses. The mean number of destructive flaws can be found by integration over the volume

$$N_{c,S}(\sigma) = \int n_c(\sigma, \boldsymbol{r})\mathrm{d}V \quad . \tag{6}$$

In general the relative frequency of flaw sizes $g(a,\boldsymbol{r})$ is unknown. Of course the flaw population is strictly related to the production process in most cases. To give some examples, typical volume flaws are large grains, pores resulting from organic inclusions, pressing defects ore remnants of agglomerates [1, 23]. Very often flaws also come into existence by the machining of the component (surface flaws) and are then strictly related to the direction of the movement of the machining tool [24]. In service inadequate handling may cause contact damage (e.g. Hertzian cracks [25], edge flakes [26]). It is obvious that a smooth and structure less frequency distribution of flaw size can hardly be expected. In the following the influence of some typical frequency distribution functions on the fracture statistics is discussed.

2.1 Weibull distribution

In addition to the assumptions made above it is assumed[3], that a material only contains homogeneous distributed volume flaws[4] with a relative flaw size density as given in Eq. 7 (continuous line in Fig. 1):

$$g(a) = g_0 \cdot \left(\frac{a}{a_0}\right)^{-r} \quad . \tag{7}$$

This function has only two parameters effectively: the exponent ($-r$) and a coefficient ($g_0 \cdot a_0^{\;r}$). g_0 and r are material parameters and a_0 is a normalizing length[5]. The integral Eq. 5 gives:

$$n_c(a_c) = \frac{a_c}{r-1} \cdot g(a_c) \quad . \tag{8}$$

For a homogeneous material the integral Eq. 6 is trivial. Using Eq. 4 holds: $N_{c,S}(\sigma) = Vn_c(\sigma) \propto V \cdot \sigma^{2(r-1)}$. By comparing Eq. 1 and Eq. 2 results:

$$m = 2(r-1) \quad . \tag{9}$$

[3] A similar derivation can be found in many papers. For more details see [13, 27].
[4] Analogue calculations could be made for surface cracks or edge cracks.
[5] A material with a relative frequency of flaw sizes according to Eq. 7 is called Weibull material.

The same behaviour can also be observed for brittle materials having flaws populations with any orientation [13].

In summary, a Weibull distribution of strength can be expected for ceramic materials containing sparsely distributed flaws, which have a relative frequency of sizes which follows an inverse power law[6]. Such "Weibull behaviour" can be expected – at least in a limited interval of flaw sizes – for many advanced ceramics.

Weibull statistics is also applied to describe the behaviour of specimens and components, which are loaded – in general – without a uniaxial tensile stress but with a locally varying stress field. In this case the relationships determined for tensile testing remain valid, if the stress in Eq. 1 is replaced by a suitable equivalent stress (to account for the action of multi axial stress states,) and the volume is replaced by the effective volume[7] [7]. Details can be found in standard text books [2-4].

The relative flaw size density can be determined from a known Weibull statistics (Eq. 1) in the following way: For a homogeneous material the density of critical defects is $n_c(\sigma) = N_c(\sigma)/V$. Using Eq. 1 and Eq. 2 this expression can be transformed into $n_c(\sigma) = \frac{1}{V_0} \cdot \left(\frac{\sigma}{\sigma_0} \right)^m$. Inserting into Eq. 8 and using the Griffith/Irwin failure criterion (Eq. 4) and Eq. 9 results:

$$g(a) = \frac{m}{2V_0} \cdot \left(\frac{K_{Ic}}{Y\sigma_0\sqrt{\pi}} \right)^m \cdot a^{-r} \quad . \qquad (10)$$

That expression can be used to identify the material properties presented in Eq. 7 (Remember that *m* = 2 (*r*-1)).

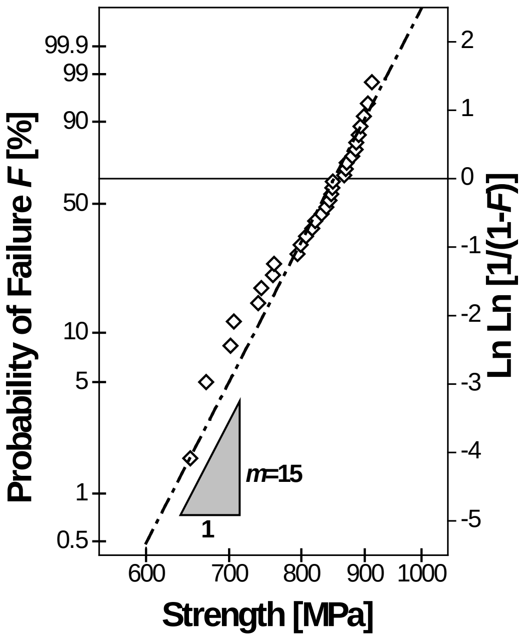
Fig. 2.a gives an example for an experimentally determined Weibull distribution. Shown are standardised 4-point bending tests [10] on a commercial silicon nitride ceramic. The material and the tests are described in [28]. The tests have been performed according to ENV 843-1 [10] and evaluated according to ENV 843-5 [11]. The data are plotted in such a way, that a Weibull distribution is represented by a straight line [7]. It can be recognised that the 30 data points are nicely grouped around the Weibull line. Typical fracture origins are glassy regions, which are - in many cases – closely related to iron rich inclusions, Fig. 2b.

The Weibull modulus of the sample (containing 30 tests) is *m* = 15 and the corresponding material parameter *r* = 8.5. The lowest measured strength value is σ_f = 651 MPa and the

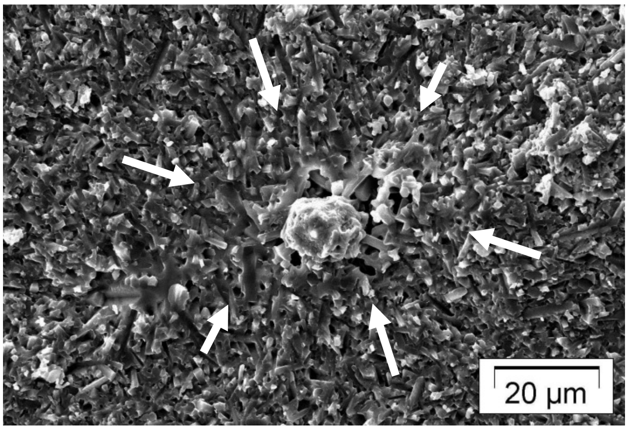
[6] The same behaviour can also be observed for materials with arbitrary oriented flaws [13].
[7] The effective volume of a component is the volume of a hypothetical tensile specimen loaded with the maximum equivalent stress amplitude of the component and giving the same probability of failure.

highest is $\sigma_f = 905$ MPa. With $K_{Ic} = 4.9$ MPa$\sqrt{m}$ (determined using the SEVNB method [29, 30]) and using $Y = 2/\pi$ (the geometric factor of a small penny shaped volume crack) the size of the fracture origin can be determined using Eq.4: they are 45 µm and 23 µm respectively. Fig 2.c shows the related relative flaw size density (Eq. 10) and the data points, which correspond to the Weibull representation in Fig. 2.a.

(a)



(b)



(c)

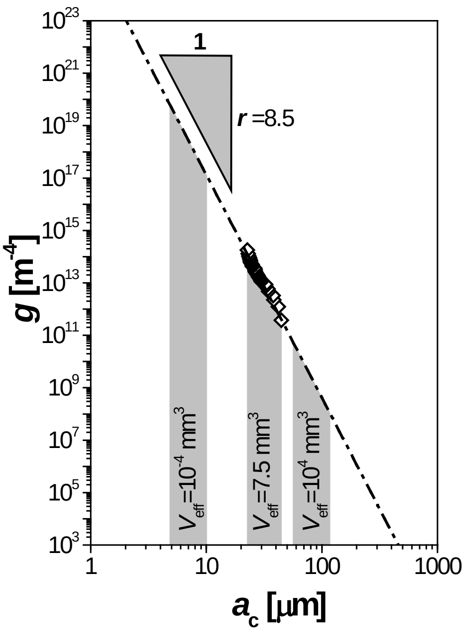


Fig. 2: 4-point bending strength test results on a silicon nitride material: (a) Probability of failure versus strength in a Weibull diagram: the strength data are aligned along straight line and are therefore Weibull distributed, (b) fracture origins are glassy regions, which are often related to iron inclusions, the arrows mark the boundary of the glassy region; and (c) relationship between the relative frequency density of flaw sizes and the flaw size. The effective volume of the specimens is 7.5 mm^3. Also indicated are the expected regions for the occurrence of fracture origins for one sample containing 30 specimens having a much larger volume and another sample containing 30 specimens having a much smaller effective volume.

A method to determine the data points can be found in [27]). It can be recognised that due to the high exponent (r = 8.5) the variation in the density is higher than two orders of magnitude. The expected interval of the sizes of fracture origins for sets of 30 specimens with an effective volume of $V_{eff} = 10^4$ mm^3 and $V_{eff} = 10^{-4}$ mm respectively are also indicated in Fig. 2c. For larger specimens, the interval of the sizes of the fracture origins shifts to the right and for smaller specimens to the left. The width of the interval is determined by the sample size.

This simple evaluation shows, that the Weibull distribution describes the situation in materials with a special (but often occurring; at least in a limited range of parameters) flaw size distribution. If Weibull statistics is used to calculate a tolerable strength for a low failure probability, extrapolations out of the empirically investigated parameter range has to be made (to the densities of larger flaws).

2.2 Unimodal flaw size distributions

It is clear from the preceding sections that the fracture statistics of brittle materials is strongly related to the flaw population. The Weibull distribution is the most prominent example for a unimodal distribution. But it can only be valid at intermediate flaw sizes. At one hand side for $a \geq L$ (L being the "size" of the specimen) it holds $g(a) > 0$ and the same holds for the number of destructive flaws. On the other hand side for $a \to 0$ holds $g(a) \to \infty$. Both are not realistic assumptions.

There exist also other limitations of the Weibull theory. From Eq. 1 it follows that the strength of a sample increases with decreasing sample size. For the same probability of failure it holds:

$$V_1 \sigma_1{}^m = V_2 \sigma_2{}^m \tag{11}$$

Therefore – in the mean – the critical flaw size is smaller in small specimens than in large ones and the density of critical flaws in small specimens is higher. In a Weibull material the relative frequency density may even be so high that the flaws interact. Then the weakest link theory is not longer applicable [27]. It can be assumed that under the action of tensile stresses the flaws grow together and form larger flaws, which limit the strength of small specimens [27]. A similar behaviour is expected to occur in brittle porous materials.

A different behaviour occurs for materials containing flaws of a single size (or if their size is narrow peaked, Fig. 3.a) Then the probability of failure is a step function of the stress amplitude. It is zero ($F = 0$) for $\sigma < \sigma_c$ and it is $F = 1 - \exp(-V n_c)$ for $\sigma \geq \sigma_c$ Fig. 3.b). The critical stress σ_c is the failure stress for a specimen containing flaws according to the failure criterion (Eq. 3) and n_c is the density of critical flaws. It is obvious that the probability of failure is smaller than one, even for very high stresses and for specimens with a very high volume. This reflects the fact that in the theory of brittle fracture only brittle failure from sparsely distributed flaws is taken into account and that there remains always some probability that a single specimen does not contain one critical flaw. Hunt and McCarthy [16] discussed that case and claimed that at high enough stresses other failure modes (e.g. plasticity induced failure) must come into existence. This case is also indicated in Fig. 3.b (vertical dotted line).

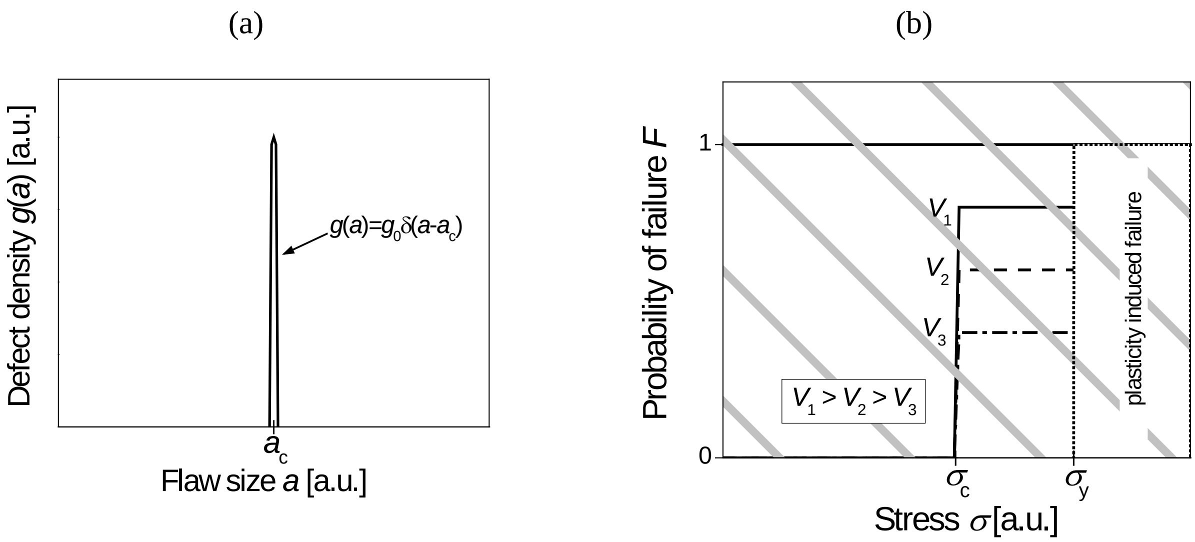


Fig. 3: Narrow peaked (delta function) flaw distribution: (a) relative flaw size density (density of defects: $g_0 = 5\cdot10^{10}$ [m^{-4}]; $a_0 = 26.1$ [µm]). (b) Corresponding fracture statistics. The probability of finding a flaw in a specimen depends on the specimen size and the same does the probability of failure. Note that the probability of failure does not reach one in any case. But if the stresses get to high additional failure mechanisms become important (e.g. plasticity induced failure), which limit the strength of the material. Then the probability of failure is one at stresses higher than the yield strength σ_y .

2.3 Bi- and multi modal flaw size distributions

Flaws are inhomogeneities in the microstructure, which result from the processing, the machining or the handling of the specimens. Examples in ceramic materials are inorganic (Fig. 2.b) or organic inclusions, hard or hollow agglomerates, badly sintered grain boundaries, large grains or cracks arising from the machining [23]. It is obvious that each individual flaw population will have its typical size distribution. This has strict consequences on the fracture statistics. This aspect will be discussed with two examples.

The first example occurs quite often when testing brittle materials. In a sample of specimens made from a brittle material two types of fracture origins are found frequently: surface flaws (arising from an inadequate machining) and volume flaws (arising from the material processing). The number of critical flaws per specimens is $N_{c,S}(\sigma) = N_{c,surface}(\sigma) + N_{c,volume}(\sigma)$ and each population can be evaluated separately. An example for the resulting fracture statistics is shown in Fig. 4. For the calculation it has been assumed that the volume as well as the surface flaws are Weibull distributed. For the volume flaws the population of the 4-point bending test data shown in Fig. 2 is used. For the surface flaws the Weibull modulus is chosen to be $m = 7$. For the relationship describing the specimens with intermediate size (standard bending test specimens) it is assumed, that at $\sigma = \sigma_0$ and $V = V_0$ the critical surface flaws occur as frequent as the critical volume flaws. The probability range observable after testing 30 specimens is shaded. In the upper part of this range the volume flaws cause a bend in the strength distribution. The curve concerning the largest specimen (smallest specimen) describes the behaviour of bending specimens, which are linearly blown up (scaled down). The effective volume is determined in the usual way [3, 4] and the effective surface corresponds to the surface area underneath the inner loading rolls. Note that for a sample of 30 specimens and for the largest specimen only surface flaws are responsible for fracture ($m =$ 7), but for the smallest specimen only volume flaws cause fracture ($m =$ 15). This behaviour is often observed in the daily testing practice [24]. The position of the structures in the fracture statistics (kink) depends on the stress as well as on the size of the specimen.

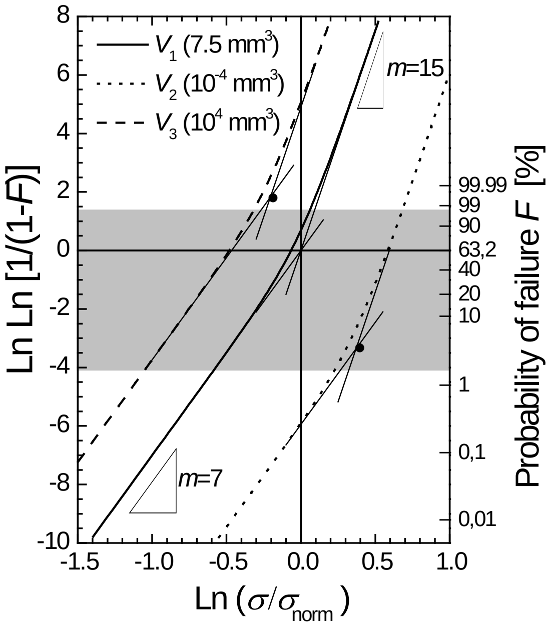


Fig. 4: Fracture statistics of samples containing volume defects (steep straight lines) as well as surface flaws (less steep straight lines). The probability range recognised by a sample of 30 specimens is shaded. It should be mentioned that, in this case, for small specimens only volume flaws and for large specimens only surface flaws are relevant for failure. For an intermediate specimen size surface as well as volume flaws may cause failure.

The second example reflects the behaviour of a material containing two different volume flaw populations, the first one (I) corresponds to the population described in Fig. 2b and the second one (II) is the narrowed peaked population of Fig. 3.a. The number of critical flaws per specimens is: $N_{c,S}(\sigma) = N_{c,I}(\sigma) + N_{c,II}(\sigma)$. The first population causes a Weibull distribution (Fig. 2.a) and the second a step distribution, Fig. 3.b. The sum of both populations is shown in Fig. 5.a. Also shown is the number of critical defects versus the defect size. The narrow peaked population causes a knee like structure in the density of critical defects, which is also mapped in the fracture statistics. For the largest specimens only volume flaws are fracture origins in a sample of 30 specimens, for the specimens of intermediate size, both populations may cause failure and for the smallest specimens only the population II is relevant for failure (the corresponding Weibull modulus would be infinite). Note that the Weibull modulus (the slope on the curves in the Weibull diagram) depends on the applied stress.

(a) (b)

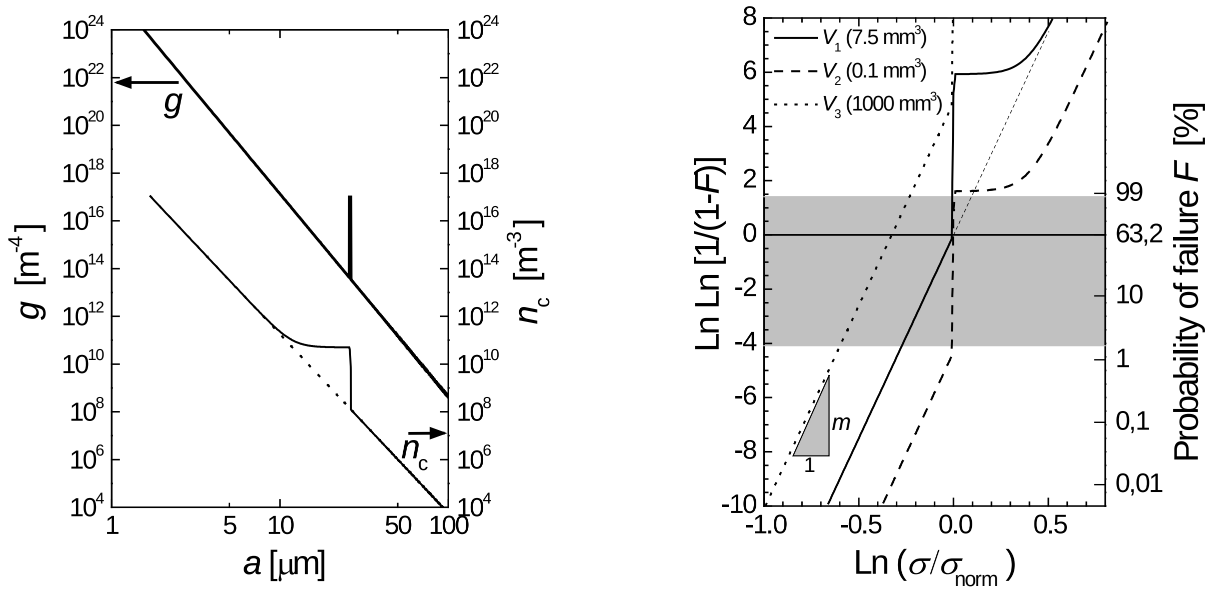


Fig. 5: Example for a bimodal fracture statistics: (a) relative frequency of flaw sizes and density of critical flaws. (b) Probability of failure versus strength. Different curves refer to different specimen sizes. The shaded area corresponds to the probability range observable after testing 30 specimens.

Analogue results can be expected for other types of bi- modal and multi- modal flaw populations. Structures in the strength distribution appear due to structures in the flaw size distribution and vice versa. They occur in a typical stress interval and do not depend on the specimen size.

It should be noted that in both discussed cases in this section the strength distribution is not a Weibull distribution.

2.4 Ceramics with increasing crack resistance curve

If the fracture toughness increases after certain crack extension (R-curve behaviour) some stable crack growth $a \rightarrow a_0 + \Delta a$ before fracture is possible. Then the Griffith/Irwin fracture criterion has to be supplemented by the condition $\partial K / \partial a \geq \partial R / \partial a$ as demonstrated by fracture mechanics. (R being the crack resistance which depends on the crack extension Δa: $K_{Ic} \rightarrow R = K_{Ic,0} + \Delta K_{Ic}$). In this case the strength depends not only on the initial toughness and flaw size but also on the increase of toughness and crack length: $\sigma_f = \left(K_{Ic,0} + \Delta K_{Ic}\right) / \left(\pi\left(a_0 + \Delta a\right)\right)^{1/2}$. The increase in fracture toughness, ΔK_{Ic} as well as the crack extension, Δa, depend on the shape of the R-curve but also on the size of the initial flaw, a_0. For details see standard textbooks, e.g. [3, 4].

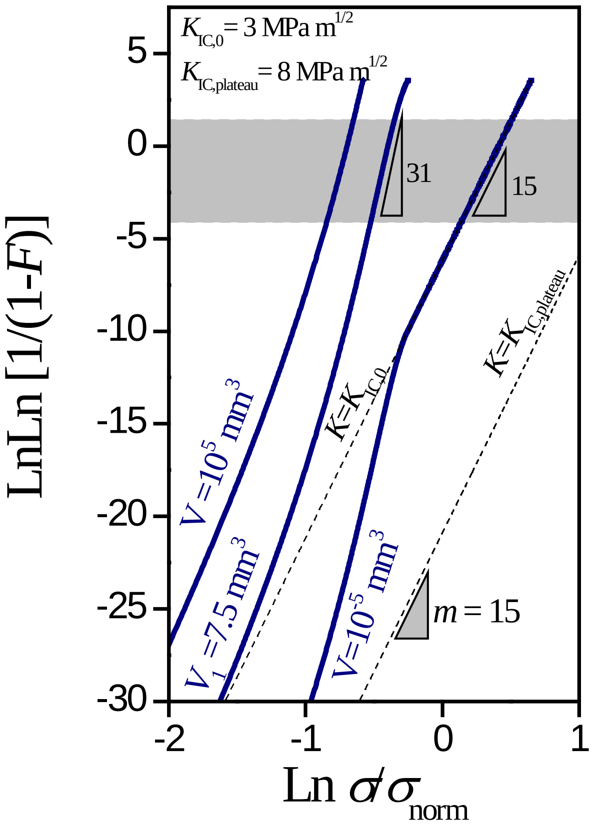


Fig. 6: Probability of failure versus strength in a Weibull plot for a material with a pronounced R-curve behaviour: The three solid lines refer to specimens with different volume. The hatched area is the range observable with 30 strength tests. At low (not showed in the plot) and high failure stresses the curves are straight, which indicates a "Weibull behaviour", but in the intermediate stress range, the curves are bent. This causes a stress dependent Weibull modulus and the fracture statistics has a non Weibull behaviour. The steep slope of the curves at intermediate stresses indicates a restricted scatter of strength dated in that range.

The consequences of such behaviour for fracture statistics are shown in Fig. 6, where the probability of failure is plotted versus strength in a Weibull diagram[8]. The R-curve is selected in such a way, that it corresponds to R-curves of published for silicon nitride materials [31]. The defect population is that of a Weibull material ($m = 15$). The bold curve approaches the dotted Weibull line (for $R = K_{Ic,0}$) at the high stress side and the Weibull line (for $R = K_{c,plateau}$) at the low stress side of the distribution. At intermediate stresses, this distribution moves from one line to the other. This results in a reduced scatter of strength data (increased modulus) in this parameter range and causes again a stress dependent modulus. In the used example it reaches its maximum ($m = 31$) in the probability range, which is observable with 30 specimens (shaded area). For specimens with a much larger volume only the plateau region of the fracture toughness is relevant in the plotted stress range. This is beneficial for the strength (see Eq. 3) but has almost no influence on the scatter of the strength data (on the modulus). Again structures in the statistics can be found in the same stress region and do not depend on the specimen size. Again, the strength is not Weibull distributed

2.5 Other effects influencing the Weibull modulus and the strength distribution

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[8] Data used for the construction of the diagram: initial fracture toughness $K_{Ic,0}$ = 3 MPa·m$^{1/2}$; the R-curve increases to the plateau value $K_{Ic,plateau}$ = 8 MPa·m$^{1/2}$ according to $R = K_{Ic,plat} - (K_{Ic,plat} - K_{Ic,0})\, e^{-(\Delta a/\lambda)}$ .The characteristic length λ is 100 µm. The initial defect population is that shown in Fig. 2.c

There exist many other effects, which may affect the shape of the strength distribution, which cause a strength and volume dependent Weibull modulus or, in other words, cause a “non-Weibull” strength distribution. Some examples are mentioned in the following.

A simple example is the case of specimens containing internal stresses. Then the internal stresses have to be added up with the applied stresses. For internal compressive stresses (e.g. surface stresses, which are relevant in bending testing) this can cause a threshold stress for failure. This may lead to the behaviour described by the tree parameter Weibull distribution. In general internal stresses cause a stress dependent Weibull modulus and lead therefore to a “non-Weibull” strength distribution. Similar effects are also claimed for materials with a non-homogeneous microstructure by Danzer et al. [19].

Fett et al. [32] analyzed the case of steep stress gradients, as they occur due to contact loading or thermal shock loading. In this case the stress amplitude is not longer (approximately) constant over the crack length. This has strong influence on the stress intensity factor K, which in the case analyzed by Fett et al. [32], gets proportional to the crack length. In consequence and in a local scale, the “Weibull modulus” m gets approximately proportional to the parameter r, which describes the distribution density of the initial flaw sizes.

Zimmerman [33] analysed the case of a crack in front of a pore (such defects are often observed in ceramics produced by pressing and sintering spray dried powders; in this case the pore is covered by many small cracks, their size is of the order of the size of the grains). For that case (pore diameter is large compared to the grain diameter) the action of the pore is a rise of about three times the applied stress but the size of the pore has no longer an influence on the strength. Under these conditions the strength depends on the size of the crack in front of the pore, i.e. – in general – the size of the surrounding grains and no longer on the size of the pore. In consequence the size effect on strength disappears. Similar behaviour is observed in materials containing flaws of such high densities that interaction between flaws gets possible [34].

2.6 Size effect on strength

As mentioned in the introduction the size effect on strength, which is described by Eq. 11, is one of the must important features of the fracture statistics of brittle materials.

2.6.1 Weibull material

For the strength data shown in Fig. 2 (standard 4-point bending tests on standard Si_3N_4 specimens), the consequences are shown in Fig. 7.a. Plotted is the characteristic strength of the sample σ_0 versus the effective volume of the specimens in a double logarithmic scale. The expected change of the strength with the volume predicted by Eq. 11 is shown by the bold line. The dashed lines give the 90 % confidence intervals for the Weibull strength prediction.

Also shown are further bending test results of samples containing specimens with smaller and larger effective volume[9]. It can be recognised, that they are within the 90 % confidence limits of the predicted behaviour. This fact confines their consistency with the Weibull theory. In this case the Weibull modulus does not depend on the stress, i.e. in the investigated range of parameters; the material is a Weibull material.

But it can be also recognised that the measured strength values are on the borders of the prediction limits. The data would fit better to a straight line with slope $-1/m = -1/20$. In fact the mean modulus of the other three samples is around 19. It can be concluded that a data fit on the basis of the size effect give more reliable results than a fit to an individual sample of a Weibull distribution. Such a procedure can be carried out with very small sub-samples for the individual specimen sizes, since the mean (characteristic) strength of the sample can be determined with a few tests (approximately 10). The typical critical flaw size is related to the specimen size. By testing different volumes a spread of the flaw size interval is reached (see Fig. 2.c) and the determination of the flaw size density can be done on a wider base.

2.6.2 Bimodal strength distribution

In Fig. 7.b the size effect on strength is demonstrated for the bimodal distribution of Fig. 5. The narrow peaked flaw population II has only influence on the strength in a limited interval of specimen sizes and stresses (in this interval the volumes vary four orders of magnitude). In this range the strength may even be independent of the specimen size. Outside of this range the population II is not relevant and fracture is triggered by flaws from population I, which is that of a Weibull material. The dashed line shows the behaviour of a material, which only contains flaws of type I.

The effective volumes of the specimens discussed in Fig. 5 are also indicated in Fig. 7.b. It can be recognised that for specimens with the effective volume V_1 or V_2 the characteristic strength depends on the flaws of population (II) and for specimens with the effective volume V_3 it depends on the flaws (I).

This example shows that deviations from the Weibull behaviour can be read off a strength volume diagram for the same reason as discussed in the last paragraph.

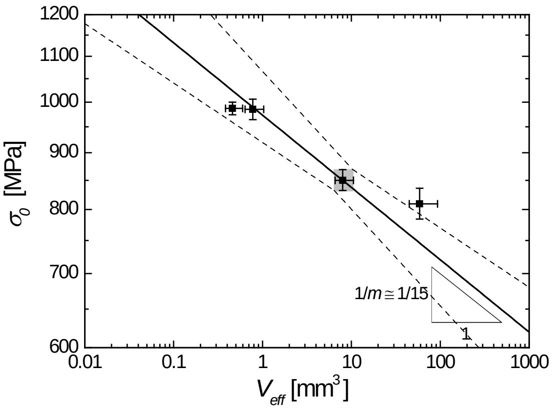
2.6.3 R curve behaviour

Fig 7.c shows the size effect on strength of a material with R-curve behaviour (Fig. 6). Again a deviation of the Weibull behaviour occurs, which – in this case – is a consequence of the controlled crack extension and the related increase in toughness. This effect depends on the initial flaw size. As discussed before that fact relates the strength and the volume of the specimen. Again the material behaves like a Weibull material at very small volumes (where no stable crack extension occurs and the toughness keeps its original value). For very large volumes the materials toughness reaches asymptotically the plateau value and again the fracture statistics looks like a Weibull statistics. It should be noted that the variation of volumes in the diagram is extremely wide. Such a wide variation can not occur in technical applications. For a realistic range of volumes the discussed example causes an increased

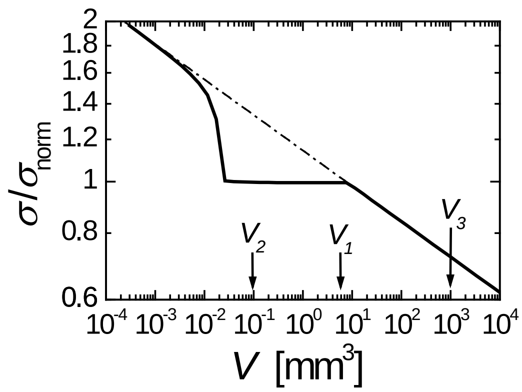
[9] The span lengths corresponding to the smallest effective volume are 13 and 4.33 mm, for the largest effective volume are 80 and 40 mm

"apparent modulus" (around 30). This corresponds to a reduced scatter of strength data and is relevant in the design of more reliable materials.
In summary, testing specimens of different volumes helps to understand the statistical behaviour of brittle materials. The main advantage to the usual practice of testing samples containing specimens of a unique size is the possibility to collect information about critical defects in different sizes range with a limited experimental effort (see Fig 2.b).

(a)



(b)



(c)

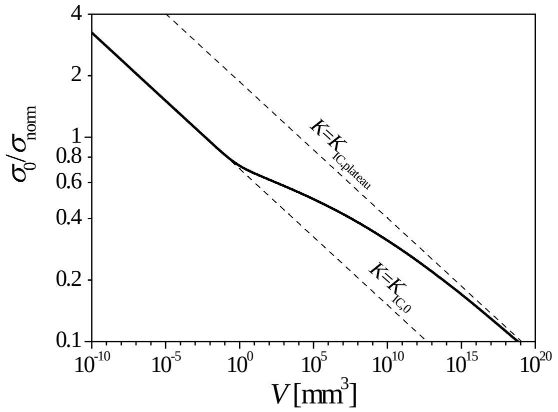


Fig. 7: Strength versus specimen's volume in a double logarithmic plot for (a) the Weibull material (shown in Fig. 2), (b) a material with a bimodal flaw population (shown in Fig. 5) and (c) a material with an R-curve behaviour (shown in Fig. 6). In part (a) experimental bending test results, which are consistent with the Weibull theory, are also plotted.

3. Monte Carlo simulations – sample and population

In the above paragraphs many arguments were given that the fracture statistics of a brittle material should – in general – not follow a Weibull statistics. But it is claimed in almost any experimental work on the strength of ceramics, that the strength of a sample is Weibull distributed. Arguments to clarify this discrepancy will be given in the following.

In the following a Monte Carlo simulation technique is used to simulate virtual experiments to judge how precise a sample (of 30 specimens) can describe the whole population: Following the concept of Monte Carlo, random numbers between 0 and 1 are generated. The number of random numbers is equal to the size of the sample. Then - for a given strength distribution - the corresponding strength values are determined (via virtual strength testing, [20]). In this way samples of the population can be easily determined. The great advantage of this procedure is, that on the basis of a precisely known population the behaviour of samples can be studied.

Fig. 8 shows some virtual strength distributions, which result from the Monte Carlo procedure described above. The used population is the fracture statistics shown in Fig. 2. Following the instructions of the standards, e.g. [11], the sample size is $N = 30$. Of course, some of the samples show the Weibull behaviour corresponding to the population (Fig. 8.a), but many of them look like bimodal or other non-Weibull distributions. A classification in different types is shown in Fig. 8. The examples are selected out of 150 sample simulations. Although the differentiation between different types is a little diffuse, Table 1 presents a rough impression about the relationship between sample and the whole population. Since the distribution of type *c* and *e* look quite similar and was relatively rare, they have been put into a single class.

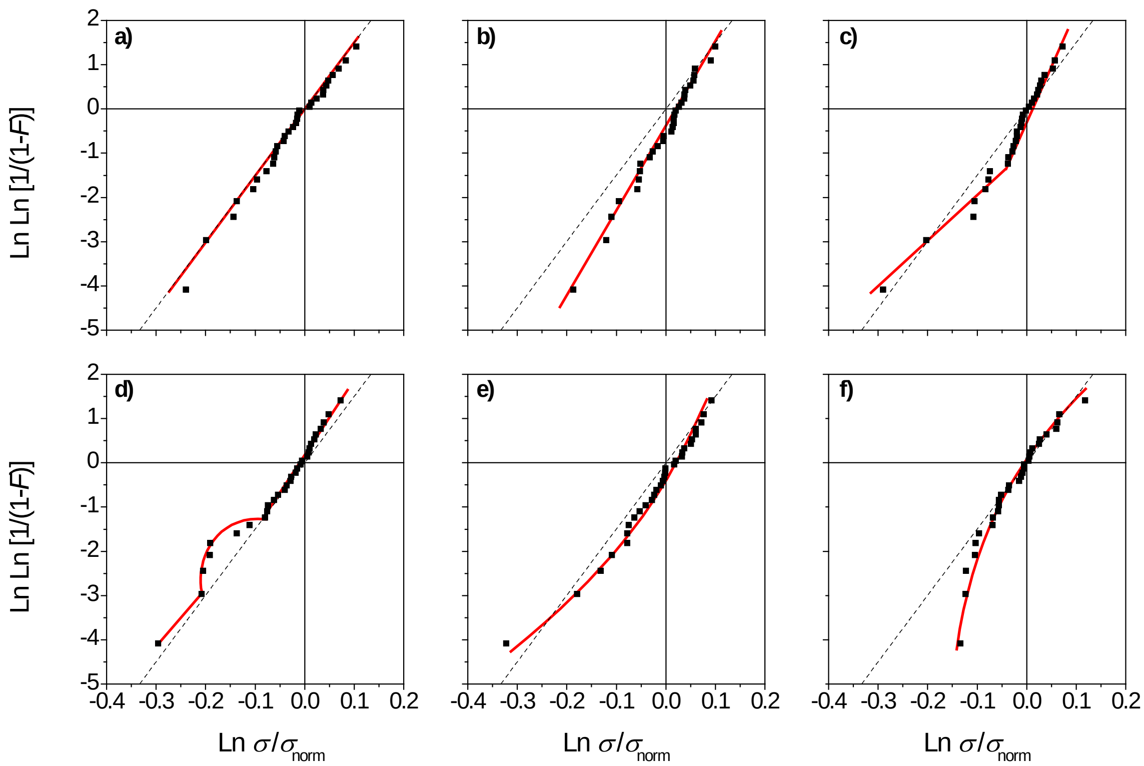


Fig. 8: Strength distributions generated by Monte Carlo simulations. The dashed line indicates the origin population and the solid lines indicate different types of samples: (*a*) the sample represents the population quite well, (*b*) the sample fits a Weibull distribution, but has a quite different modulus and/or mean strength, (*c*) the sample seems to follow a bimodal distribution of Fig. 4, (*d*) the sample could fit a bimodal distribution as shown in Fig. 5, (*e*) the sample seems the distribution of an R-curve material as shown in Fig. 6, (*f*) the sample fits a three parameter distribution occurring due to internal compressive stresses.

The authors have also done Monte Carlo simulations of samples produced from the non Weibull distributions discussed above (bimodal, etc.). It was not surprising, that in any case many samples could be found which looked like a Weibull distribution. This problem is also discussed in some of the papers of Lu et al [9, 35]. In the relevant parameter range a clear distinction between different statistical distribution functions would make the testing of several thousand specimens necessary [9]. Under these conditions it is clear that a Weibull distribution can be adjusted to any small sample of a fracture statistics of a brittle material. But it should be recognised, that this does not necessarily mean, that the population is a Weibull distribution.

Table 1. Classification of samples according to the types shown in Fig. 8.

Sample class	*a* Weibull $m^* \approx m$ [a]	*b* Weibull $m^* \neq m$	*d* bimodal narrow peak	*f* compressive internal stress	*c + e* R-curve or bimodal wide populations
estimated fraction [%]	25 – 35	15 - 25	20 - 30	10 - 20	5 - 15

[a] where m^* is the modulus of the sample and m the modulus of the population

4. Summary and conclusions

The fracture statistics of brittle materials reflects the size distribution of flaws in brittle materials. A Weibull statistics only occurs under special conditions, which are not valid in general. Especially a Weibull distribution is not expected to occur for brittle materials containing bi- or multimodal flaw distributions, surface and volume flaws, having an R-curve behaviour, showing internal residual stress fields, or having a high defect density. It is also not expected to occur if very small specimens are tested or if the applied stress field presents high gradients. In all these cases the Weibull modulus is no longer a constant and depends on the applied stress amplitude.

The strength distribution is a mapping of the flaw size distribution. The size (volume) of the specimen defines the mean strength and therefore also the mean size of the mapped critical flaw. The size of the sample defines the width of the mapped flaw interval. This width is relatively small for standardized sample sizes (e.g. $N = 30$). Therefore the fitting of a strength distribution to data in that (small) interval is relatively unreliable (e.g. for a set of 30 strength tests the typical width of the 90 % confidence interval of m is ± 30 %). The reliability of the determination of the strength distribution can be improved if the width of the mapped flaw size interval is increased. This can be done by testing specimens of different size (i.e. effective volume). Then the determination of the strength statistics (Weibull parameters) is much more reliable.

Published strength data based on one single sample are generally claimed to be Weibull distributed but this needs not be true. In general, they are determined on small samples since the machining of specimens is very expensive. On the basis of a small sample it is not possible to decide whether a distribution is a Weibullian or not.

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