Lehrstuhl für Angewandte Mathematik

Dissertation

Equilibrium Models for Asset Bubbles

A Regime Switching and a Lévy Model Approach

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Leoben, Oktober 2018

Affidavit

Declaration on oath

Boulogne-Billancourt, 23rd October 2018

I declare in lieu of oath, that I wrote this thesis and performed the associated research myself, using only literature cited in this volume.

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Boulogne-Billancourt, den 23.Oktober 2018

Ich erkläre an Eides statt, dass ich diese Arbeit selbständig verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt und mich auch sonst keiner unerlaubten Hilfsmittel bedient habe.

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Abstract

An asset bubble, also referred to as a speculative mania or financial bubble, is an economic situation characterised by trading in an asset at a price that is far above its real value. This is often followed by a sharp drop which is known as bubble burst. As such, a large amount of wealth can be destroyed, and this may lead to a continuing economic crisis. Therefore, asset bubbles have become a subject of growing interest in mathematical finance. There are several approaches towards this subject which can be divided into two main groups: models that quantify price bubbles in a classical arbitrage-free setting and models that explain the mechanism of price bubbles. In this work, price bubbles are introduced as the difference between a minimal equilibrium price and an intrinsic value. The aim of this study was to find out how we can include sudden changes in the underlying or the economic situation and what could be its possible effect on an asset bubble. Our first approach was to introduce Markovian regime switching in the interest rate. We revealed that the bubble contains a component that is entirely based on the regime switching risk. The second approach was a Lévy model for the underlying asset. Our findings in this part provide a basis for further research on the current topic with numerical implementation.

Zusammenfassung

Unter einer Preisblase, auch Spekulations- oder Finanzblase genannt, versteht man eine Situation, in der Finanzgüter zu Preisen, die weit über deren tatsächlichem Wert liegen, gehandelt werden. Dem folgt oft ein abrupter Kurssturz, der als Platzen der Blase bezeichnet wird. Auf diese Weise können große Vermögen vernichtet und in Folge lang anhaltendes wirtschaftliches Chaos ausgelöst werden. Aus diesem Grund wuchs das Interesse in Preisblasen für finanzmathematische Forschung in letzter Zeit stetig. Es gibt verschiedene Zugangsweisen zu diesem Thema, die sich in zwei große Gruppen unterteilen lassen: Modelle im Kontext klassischer, arbitragefreier Marktmodelle, die Blasen quantifizieren, und Modelle, die versuchen, den Mechanismus dahinter zu erklären. In dieser Arbeit bezeichnen wir den Unterschied zwischen minimalem Gleichgewichtspreis und Sachwert als Blase. Das Ziel war zu erklären, wie sich plötzliche Änderungen im Kurs des Finanzguts oder in der globalen wirtschaftlichen Situation auf mögliche Preisblasen auswirken. Unsere erste Herangehensweise war Markovsches Regime Switching im Zinssatz. Dabei konnten wir zeigen, dass ein Teil der Blase ausschließlich durch die Möglichkeit, das wirtschaftliche Regime zu ändern, erklärt wird. Die zweite Herangehensweise war ein Lévymodell für das Finanzgut. Unsere Ergebnisse legen den Grundstock für weitere Forschung zur numerischen Umsetzung.

Dedicated to my Grandmother

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Contents

1. Introduction

1.1. Structure and Methodology

Beginning with a general review of asset bubbles, we put them into historical and economic context. In the following two sections, we intend to give an overview of the existing mathematical literature on asset bubbles focussing on more recent work. The two large groups, the classical semimartingale approach and the equilibrium models, are in the centre of our discussion. In the following chapter, we summarise regime switching, a field that has recently become particularly interesting. Next, we recapitulate the model of Chen and Kohn [21] which will be subsequently generalised to a regime switching environment as well as a Lévy setting.

The Preliminaries, described here, are for better understanding the mathematical basis for the present work. First, we highlight the main concepts and properties of Lévy processes and stability. Since there is no unique notation for them in literature, this chapter helps to avoid ambiguities. Changing the measure is an essential step in all of our models and therefore the Girsanov theorem will be mentioned. The following two sections are used to introduce pseudodifferential equations and viscosity solutions. The proofs can be found in the concerning references.

Afterwards, in Chapter 3, we modify Chen and Kohn's model by replacing the Ornstein Uhlenbeck process with a square root diffusion process. The results allow us to understand how every part of the theory is affected.

In Chapter 4, we set up a regime switching model for asset bubbles. Thereby, we combine the approach from Chen and Kohn [21] with a classical regime switching setting [36]. The switching affects only the interest rate for following reason: The model interest rate is driven by a general economic situation that can be changed; The other model parameters, however, remain constant over the time. In other words, the gap between the investors' opinion stays always the same. The dividend rate is the only source of income from the asset and is given by an Ornstein-Uhlenbeck process. There are two investors that disagree on only the mean-reversion-rate; they chose different measures. Showing the equivalence between these measures is the first technical challenge. Here, we took a similar approach as Elliott [36]. Due to the regime switching component, matrix special functions as introduced in [66] get involved. Then, we discuss the solution of a matrix differential equation which is an equilibrium price under some restrictions. With the help of the theory of viscosity solutions, we can finally show that such an equilibrium price is minimal. A discussion of the result with numeric examples rounds up this chapter. The most interesting result we received is that we can indirectly price the risk of a regime switch. The bubble contains a non-negligible component that is merely based on the probability of changing to another regime.

Chapter 5 is dedicated to a Lévy model also based on [21]. We replace the classical Ornstein-Uhlenbeck process with a Lévy driven process. Finding a general representation of the intrinsic value needs a few more assumptions, including α-stability. Our definition of equilibrium prices involves transaction fees. Strategies like immediate resale can just be optimal without transaction cost. However, the existence theorem of minimal equilibrium prices can be generalised. Finally, we set up a pseudo-differential equation including the generator of a Lévy process and show that its solution is an equilibrium price. Taking a close look, we can see that for the α-stable case and supposing time independence, our equation is in fact an integro-differential equation. Thereby, the theory of viscosity solutions has been developed for this and we can proceed in a similar way as Chen and Kohn to see that the solution of our integro-differential equation is actually a minimal equilibrium price. Hence, we reach to the existence and the bubble's behaviour, but unfortunately no explicit representation.

The appendix explains special matrix functions where the matrix confluent hypergeometric equation is solved. We studied the second Kummer matrix function and its limit behaviour in detail. This was necessary as it has not been investigated before. Since the proofs are quite long, but independent from the rest of our theoretical background, they are described in a separate place. The results are also published in [100].

The literature research was mainly done with the help of *MathSciNet* (`http://ams.u-strasbg.fr/mathscinet/`) with access provided by Technical University of Graz and University Paris Diderot (Paris 7), directly in the library catalogue and occasionally with *Google Scholar* `https://scholar.google.at/`. The later mentioned calculations were implemented and performed both in Octave and in Matlab 2016 under a licence provided by Montanuniversität Leoben on a computer with a 2.80GHz processor running under Ubuntu.

1.2. Introduction to Bubbles

Throughout history, speculation bubbles have caused crises and chaos in the world's economy. The first documented and probably best known example for this phenomenon is the Tulip mania between 1634 and 1637 (see [71] or [49] for a comprehensive analysis). At this time, tulip bulbs were often bought at exorbitantly high prices and sold at even higher amounts. "Typically, the buyer did not currently possess the cash to be delivered on the settlement date and the seller did not currently possess the bulb. Neither party intended a delivery on the settlement date; only a payment of the difference between the contract and settlement price was expected" [49, p. 544]. Eventually, the high prices could not be sustained. At an auction, people started selling their tulip bulbs at increasingly lower prices, ultimately causing a substantial drop in price. The current example explains the basic principle as to how bubbles work:

> "[...] if the reason that the price is high today is only because investors believe that the selling price will be high tomorrow – when 'fundamental' factors do not seem to justify such a price - then a bubble exists." [93]

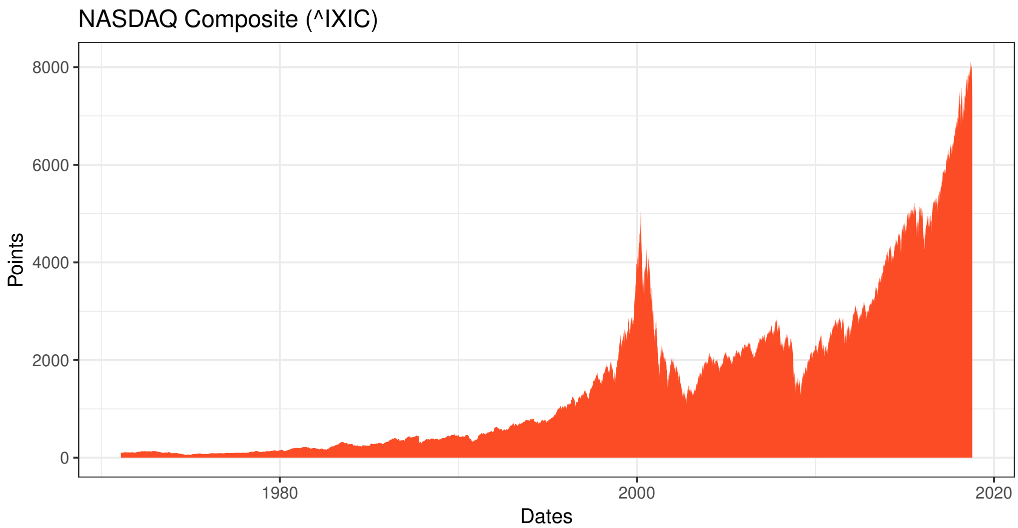

Figure 1.1.: NASDAQ Composite Index (ˆIXIC). Historical data available at `https://finance.yahoo.com/quote/%5EIXIC/history?p=%5EIXIC`.

Yet this topic is more pertinent than ever in today's economy. In 2000, the internet-bubble had its climax (see Figure 1.1) where some stocks had a growth of about 1000%. For details, please refer to Shiller's book *Irrational Exuberance* [95]. Similarly, bubbles in the uranium and rhodium market were observed in 2007 and the U.S. real estate bubble between 2006 and 2008 is widely seen as one of the main causes for the financial crisis starting in 2008. Nevertheless, it is a highly debated issue whether we are currently in a crypto-currency bubble or not. The highly volatile Bitcoin is often considered as pure speculation and seen as a bubble. In other words, one could also ask about fundamental value of Bitcoin. An empirical analysis [20] using data from 2010 to 2012 showed that "the bubble price rises are so dramatic that the estimated long-term fundamental value is not statistically different from zero.” However, the type of models used is constructed in a way that Bitcoin has no fundamental value. Deciding whether there is a bubble, highly depends on the market model and of course on the precise definition of a bubble itself. In a realistic framework, detecting a bubble can turn into to a great challenge. Therefore, a few questions rise including: How could we describe speculation bubbles mathematically? How could we explain their formation and crash in a proper mathematical way and which of those models could be used to detect speculation bubbles?

Since there is no uniform definition of asset bubbles in current literature, we distinguish them into two large main groups of models: one in a classical setting using strict semimartingales (mainly based on the work by Jarrow, Protter et al. [63]) and the other explaining the formation of bubbles via equilibria (such as the models by Scheinkman and Xiong [89] or Chen and Kohn [21]). In the following sections, we shall focus on mathematical modelling of bubbles.

1.3. The Semimartingale Approach for Asset Bubbles

Several recent papers introduced bubbles in standard market models with no-free-lunch-without-vanishing-risk (NFLVR) and no-dominance-assumption as strict local martingales. A strict local martingale is a local martingale that is not a martingale. Therefore, one distinguishes between a market price and a fundamental price. The market price - the amount to which the asset is traded - is assumed as a non-negative semimartingale $(S_t)_{t\geq 0}$ and the fundamental price denoted by S_t^* is defined as expected future cash flows under a risk neutral measure. Then, an asset bubble can be defined as the difference between fundamental and market price

$$\beta_t = S_t - S_t^*.$$

These models merely describe asset bubbles, yet they do not provide an explanation where bubbles come from. The theory was developed by the work of **R.Jarrow** and **P.Protter**. Their articles *Asset Price Bubbles in Complete Markets* [63] and *Asset Price Bubbles in Incomplete Markets* [64] became the fundament of a theory to which we will give a short introduction. Due to their contribute also to further research [60,62–64,84], their notation and their terminology has become a standard in literature. Moreover, it is consistent with classical models from financial mathematics like Black-Scholes. In *A Mathematical Theory of Financial Bubbles* [84], Protter gives a comprehensive survey of the theory and discusses points that have been criticised.

First, we need to make the notation precise. On a filtered complete probability space $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, the model consists in a money market account and a risky asset. Let a stopping time τ represent the maturity. The cumulative dividend process is assumed to be a non-negative, càdlàg semimartingale $D = (D_t)_{0\leq t\leq \tau}$ adapted to $\mathcal{F}$. The non-negative $X_\tau \in \mathcal{F}_t$ is called terminal payoff or liquidation value. The market price $S = (S_t)_{0\leq t\leq \tau}$ is assumed as a non-negative, càdlàg semimartingale adapted to $\mathcal{F}$. Precisely, for t such that $\Delta D_t > 0$, S_t denotes the price ex-dividend. Now we are able to define the wealth process as

$$W_t = S_t + \int_0^{t\wedge\tau} dD_u + X_\tau \mathbf{1}_{\tau\leq t}. \tag{1.1}$$

This plays an important role in defining the set of risk neutral measures. An Equivalent Local Martingale Measure (ELMM) is defined as a probability measure $\mathbb{Q}$ equivalent to the real world measure $\mathbb{P}$ such that the wealth process W is a $\mathbb{Q}$-local martingale and the set of ELMMs is denoted by $M^{loc}(W)$. The No-free-lunch-with-vanishing-risk (NFLVR) condition form [31] is the key no-arbitrage argument in this theory. Let $L^1(W)$ denote the space of integrable processes according to [83]. A trading strategy is defined as a pair of adapted processes (π, η) with $\pi \in L^1(W)$ such that it represents the units of the risky asset and the risk free asset respectively held at time t. The wealth process of the trading strategy (π, η) is defined by $V_t^{\pi,\eta} = \pi_t S_t + \eta_t$. A trading strategy is called self financing, if π is predictable, η optional and

$$V_t^{\pi,\eta} = \int_0^t \pi_u dW_u. \tag{1.2}$$

In other words, self-financing trading strategies start with $V_t^{\pi,\eta} = 0$ and all purchases and sales of the risky asset are financed by the money market account. Trading strategies are called admissible, if they are self financing and there exists $a \in \mathbb{R}^+$ such that $V_t^{\pi,\eta} \geq -a$

for all t almost sure. Amongst others, such strategies avoid doubling strategies and impose a lower bound to the wealth process. The NFLVR condition is, roughly spoken, excluding all self-financing strategies with zero investment that generate non-negative cashflows with positive probability, i.e. arbitrage strategies, and all other strategies approaching them. Under this assumption, the first fundamental theorem of asset pricing holds: If and only if the NFLVR hypothesis is satisfied, then there exists an ELMM. For the exact definition and the details, we refer to [31] or [63]. The second fundamental theorem also holds: In complete markets, the ELMM is unique [63]. Since the fundamental price is defined through the ELMM, it makes a huge difference, if the market is complete. Let us first focus on this case. The fundamental price is defined as the expected future dividends under the ELMM plus the payoff if the maturity is finite, or

$$S_t^* = \mathbb{E}^{\mathbb{Q}}\left(\int_t^\tau dD_u + X_\tau \mathbf{1}_{\tau \leq \infty} \middle| \mathcal{F}_t^X\right). \tag{1.3}$$

As $\mathbb{Q}$ is unique, S_t^* is also uniquely defined. Hence, we can define an asset price bubble uniquely as $\beta_t = S_t - S_t^*$. The main finding of [63] is that there are only three different types of non-trivial bubbles:

- **Type 1**: uniformly integrable martingales denoted by β_t^1,
- **Type 2**: non-uniformly integrable martingales denoted by β_t^2,
- **Type 3**: **strict local martingales** denoted by β_t^3.

and the process S_t admits a unique decomposition (Theorem 5 in [63]) into

$$S_t = S_t^* + \beta_t^1 + \beta_t^2 + \beta_t^3.$$

Further, it can be shown that

- $\beta_t^1 \to X_\infty$ a.s.
- $\beta_t^2 \to 0$ a.s.
- $\beta_t^3 \to 0$ a.s. and $\mathbb{E}\left(\beta_t^3\right) \to 0$.

This limit behaviour is very important for the interpretation of the different bubble types. A few properties directly result from the decomposition theorem: First, bubbles cannot be negative. Further, bubble maturities are always finite. The drawback of a complete market is that bubbles exist either from the beginning or simply never arise. Once they burst, the same bubble will not appear again. Another important property is that those three different types occur in different situations. To make it clearer, we repeat the examples from [63]:

- **Type 1 bubbles** exist when the asset has infinite life with a payoff at $\tau = \infty$. The represent a kind of stochastic gap between fundamental and market price. From an economic point of view, they are uninteresting.
  The classical example is the fiat money with $S_t = 1$, $\tau = \infty$, $X_\infty = 1$ and $D_t = 0$. Then, $\beta = 1$.
- **Type 2 bubbles** exist when the asset has finite life that is unbounded.
  We give a martingale bubble as example: Consider a maturity τ with $\mathbb{P}(\tau > t) > 0$ and a payoff 1 at the maturity, i.e. $S_t^* = \mathbf{1}_{t \leq \tau}$. Setting

$$\beta_t = \frac{1 - \mathbf{1}_{t \geq t}}{\mathbb{Q}(\tau \geq t)} \tag{1.4}$$

one can show that β is not uniformly integrable and $\beta_\infty = 0$ and define a market price $S_t = S_t^* + \beta_t$. This process is finite with probability 1, but the asset's life is unbounded.

- **Type 3 bubbles**: for assets whose lives are bounded.
  We take the strict local martingale bubble (from Section 2.1.1 in [28]) as an example. Set $\tau = T < \infty$, $D = 0$ and $S_t^* = \mathbf{1}_{t \leq T}$. On can show [63] that

$$\beta_t = \int_0^t \frac{\beta_t}{\sqrt{T-u}} dB_u, \tag{1.5}$$

  where B is a $\mathbb{Q}$-Brownian motion, is a strict local martingale. So, the bubble exists although the maturity is finite.

Under Merton's no-dominance assumption, there are only type 1 bubbles, but neither type 2 nor strict local martingale bubbles. Roughly spoken, no-dominance means that financial agents prefer more to less. Here, we refer to the original definition [75] or to its version in modern notation [63]. No-dominance implies NFLVR. Jarrow and Protter [63] also investigated the relation between bubbles and derivatives. The most interesting conclusion from [63] is that under the no-dominance assumption, bubbles for standard options do not occur in complete markets. The other main reason to extend the model to incomplete markets was the interest in the modelling of a "bubble birth".

To adapt the model to incomplete markets [64], the definition of the fundamental price needs to be generalised. Let $\sigma = (\sigma_k)_{k \geq 0}$ be an increasing sequence of random times with $\sigma_0 = 0$ called the shift times. Suppose σ to be independent of the current state of the economy and also of the filtration $\mathcal{F}$ to which S is adapted. Then, one can define the number of regime shifts up to t as

$$N_t = \sum_{k \geq 0} \mathbf{1}_{t \geq \sigma_k}. \tag{1.6}$$

If $N_t = i$, then $\mathbb{Q}^i \in M^{loc}(W)$ denotes the ELMM selected by the market at time t. This allows us to define the fundamental price as

$$S_t^* = \sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}^k} \left(\int_t^\tau dD_u + X_\tau \mathbf{1}_{\tau \leq \infty} \middle| \mathcal{F}_t^X \right) \mathbf{1}_{\{t \leq \tau\} \cap \{t \in [\sigma_k, \sigma_{k+1})\}}. \tag{1.7}$$

It is important to remark, that the existence of a equivalent measure $\mathbb{Q}^*$ such that

$$S_t^* = \mathbb{E}^{\mathbb{Q}^*} \left(\int_t^\tau dD_u + X_\tau \mathbf{1}_{\tau \leq \infty} \middle| \mathcal{F}_t^X \right) \mathbf{1}_{\{t \leq \tau\}}. \tag{1.8}$$

can be shown [64]. This is often called valuation measure and is an ELMM in a static market without regime shifts. However, $\mathbb{Q}^*$ is in general not a martingale measure. Therefore, the choice of the ELMM always affects the fundamental value. With some effort, [64] generalises the decomposition theorem to incomplete markets. The bubble β can be decomposed to the same three types of bubbles, but the behaviour under no-dominance changes. The bubble is no longer necessarily a martingale and bubbles can arise at some regime shifts. However, a concrete example for the "bubble birth" at a measure change was found later in a credit risk model setting [12].

Ekström and Tysk [34] discuss some aspects of the Black-Scholes model in an economy with bubbles. Several properties do not hold anymore in the new setting, such as put-call-parity or uniqueness of the solution of the Black-Scholes equation. For a continuous payoff function with at most linear growth, the risk-neutral option price is a solution of the Black-Scholes equation with at most linear growth. Further, they discuss convexity theory for European and for American options. Jarrow and Protter [61] discuss forward and futures prices in the context of asset bubbles. Jarrow, Protter and Kchia [60] focussed on statistical methods to detect bubbles and developed criteria when an asset's price exhibits a bubble. Within the standard semimartingale approach, they restrict themselves to the model

$$dS_t = \mu\left(S_t\right) dt + \sigma\left(S_t\right) dW_t \tag{1.9}$$

on a finite horizon where W denotes a Brownian motion and the function σ the volatility. Then, the market price S is a strict semimartingale if and only if

$$\int_\alpha^\infty \frac{x}{\sigma(x)^2} < \infty \tag{1.10}$$

for all $\alpha > 0$ (for a proof of this criterion see [72]). Obviously, type 3 bubbles are the most relevant in a real economic environment. If S is a strict local martingale, one can show that there is such a bubble. Therefore, by volatility estimations methods, one can determine if S has semimartingale properties and hence if there is a bubble. Finally, the internet bubble is chosen as a concrete example to illustrate their result. A recent article by Jarrow [59] gives two further ideas to detect asset bubbles: Identifying a bubble via a comparison of the asset's put and call prices and a return factor model decomposing into fundamental value and bubble. A more theoretical work by Pal and Protter [81] on h-transformations also affects this subject. They give examples for strict local martingales and put their theory in the context of option pricing. Mijatović and Urusov [78] provide a deterministic criterion for the absence of bubbles in a generalised constant-elasticity-of-variance process setting. It is one application of their theory about strict local martingale property. Jarrow, Protter and Roch [62] worked on the influence of liquidity on asset bubbles. However, their approach is slightly different. The fundamental price process is assumed to be exogenously given and the bubble endogenously determined by market trading activity due to liquidity risk. The model by Biagini, Föllmer and Nedelcu [11] slightly differs from the standard semimartingale approach. They examine a flow in the space of equivalent martingale measures. This allows bubbles to arise an disappear within shifts of the measure. Guasoni and Rásonyi [51] discuss the robustness of the local martingale diffusion models to small change like the presence of transaction cost. A paper by Kardaras, Kehrer and Nikeghbali [69] gives detailed examples of this theory's application to option pricing and discusses the relationship between risk neutral and real measure in this context. Under NFLVR, but without no-dominance assumption, they examine the influence of bubbles on the pricing of derivatives. After describing a last passage formula, the examples focus on path dependent options like European and American exchange or chooser options. The model of Biagini and Nedelcu [12] is particularly interesting, because it combines the findings from [11] with [64] and gives a concrete example for a bubble birth. Within a defaultable claim model (as discussed in [14]), they characterise the set $\mathcal{M}^{loc}(W)$ and explicitly calculate a measure change. Bilina Falafala, Jarrow and Protter [15] discuss bubbles of bounded asset as for example bond bubbles. Under a change of numéraire from the money market account to another one using the risky asset, they examine the existence of local martingale measures. Jarrow [58] developed a multiple-factor model with bubbles in an arbitrage-free, competitive, and frictionless market. A recent paper by Obayashi, Protter and Yang [80] generalises the method to detect an asset bubble from [61] to a more realistic model including stochastic interest rate. They also provide a detailed example of their method and estimation of a bubble's lifetime on real data set. The model by Herdegen and Schweizer [54] relies basically on the same concepts, but the setting has various differences. Instead of NFLVR, they take a weaker no arbitrage concept, the no unbounded profit with bounded risk (NUPBR) assumption. Moreover, their definition of a fundamental price is also substantially different. However, they can show that bubbles are strict local martingales. A different approach by Cox, Hou and Obłój [29] examines strict supermartingale bubbles in a robust derivative pricing setting. The most interesting part of this paper is the discussion about trading and definition of a fundamental price. They provide a justification of the semimartingale approach for bubbles arising from trading restrictions and market prices. Keller-Ressel's work [70] on pure-jump strict local martingales provides a new direction to construct strict local martingales by a measure change. This could be an interesting way to follow in modelling bubbles.

1.4. Equilibrium Models for Asset Bubbles

While standard market models aim to quantify and detected bubbles, equilibrium models try to explain them in different ways. Before entering in this direction, we want to ask ourselves some crucial questions: where do price bubbles come from? What mechanism lies behind them? Why do asset bubbles arise? Moreover, bubbles "are associated on occasion with general 'irrationality' or mob psychology" [71, p. 36], but since rationality is one of the key assumptions in efficient market theory and hence also in almost all mathematical models, how to include this aspect? Kindleberger distinguishes in his famous book about *Manias, Panics and Crashes* [71] into several possible explanations for irrationality in a market:

- Group thinking and herd behaviour affects all market participants.
- The degree of rationality differs among groups - traders, investors and speculators exhibit different behaviour as prices rise.
- The behaviour of a group of individuals differs from the sum of the behaviours of each of the individuals in the group.
- Individuals choose a wrong model or fail to consider crucial information.

In general, the causes that lead to a bubble are manifold: structural, cultural and psychological factors play together in a complex way. For a profound analysis of this topic, we refer to Shiller's book *Irrational Exuberance* [95]. A key notion is the term **rational bubble** which describes bubbles that exist in rational markets.

> "Possible explanations for the formation of bubbles include self-fulfilling expectations (rational bubble), mispricing of fundamentals (intrinsic rational bubble) and the endowment of irrelevant exogenous variables with asset pricing value (extrinsic rational bubble). Rational bubbles exist when investors anticipate that they can profitably sell an overvalued asset at an even higher price. In contrast, irrational bubbles are formed when investors are driven by psychological factors unrelated to the asset's fundamental value." [20]

There are a few examples of irrational bubbles. The most well-known is the so-called Ponzi-scheme, a kind of a fraudulent pyramid scheme. The manager of such a scheme promises large profits for investors, but his real investment is worth almost nothing. Investors put confidence in other market participants above their own rationality; a feedback effect for bubbles (see [95] for details) affects them. Albania's crisis in 1997 serves as an outstanding example. Several different Ponzi schemes promised their possible investors extremely large returns with always the same strategy: the main payments to old investors were financed through money contributed by new investors. The investment in such schemes went up to 30 % of the country's GDP which after the collapse of those pyramids caused a substantial economic and political crisis [95]. However, capturing irrationality from a mathematical point of view is very difficult.

Since we want to examine the origin of asset bubbles in a mathematical way, the appropriate concept of a price is indispensable. On a real market, prices are fixed through supply and demand. Each investor chooses his portfolio by maximising his expected utility subject to a wealth condition. In most models, the communication between sellers and buyers is uniquely through prices. So, the prices the seller offers have to correspond to those which the buyer accepts or it will not lead to a trade. Finding such an equilibrium price will lead to prices that are often substantially different from the fundamental value of an asset. For a detailed, precise mathematical formalisation of equilibrium theory, we recommend Barucci and Fontana's book *Financial Market Theory* [9]. They also dedicate a chapter of their book to rational asset bubbles, where they refer to all important literature.

The literature for equilibrium bubbles is harder to categorise, since there are - unlike for the semimartingale approach - many slightly different models and some branches of the theory developed in a completely different direction. Blanchard and Watson [16] investigate the consistency of bubbles with rationality in the market and provide first tests for the existence of bubbles. **Tirole** [97] is dedicated to definitions and properties of asset bubbles. He points out how important it is to precisely define fundamental prices and how sensitive bubbles are to their definition. Tirole [97] provides necessary and sufficient conditions for the existence of bubbles. An important result is the so called "no-trade theorems": private information must be excluded as a source of a bubble, since all traders act rationally and have identical prior information. Bosi and Seegmuller [18] set up a time-discrete model for rational bubbles close to Tirole's approach.

In a dynamic asset pricing model, **Santos and Woodford** [87] have given conditions such that bubbles cannot exist in a competitive equilibrium framework. A competitive market means here that every agent chooses optimal consumption under a budget constraint and under allocation of all goods. A slight modification by Werner [101] points out price bubbles can exist in equilibria with endogenous debt constraints. Loewenstein and Willard [73] study rational bubbles in a continuous setting very similar to [87] and developed conditions under which rational bubbles exist.

Further literature relies on this work or is inspired by it: Huang and Werner [56], Abreu and Brunnermeier [2], Hellwig and Lorenzoni [53] Cheriyan and Kleygweit [23] in an experimental setting, Bidian [13], Bosi, Le Van and Pham [17] in a multi-sector model, Hirano and Yanagawa [55] in an endogenous growth model and many others. In a more complex setting, Hugonnier [57] shows that portfolio constraints can create rational bubbles. Miao, Wang and Xu [77] set up a dynamic general equilibrium model. Self-fulfilling beliefs and a positive feedback loop create the bubble. There is an interesting time-discrete model [4] in which the investors' opinion on the future dividend is heterogeneous. They show that there are only three possible situations in equilibria: one investor will possess the entire wealth after some time, the return of the risky and the risk-free asset are equal and several investors coexist or where many investors share the total wealth. So, it can come to the case that one investor will drive out the others of the market. The expectations of the surviving investor can push up the price and create a bubble that will never break. Equilibria can also be defined and studied in a more complex economy that distinguishes between households, final goods producers, capital goods producers and financial intermediaries [76]. Based on that fact that bubbles occur mostly in one sector (as several housing bubbles or the internet bubble), they intensively study the structure and impact of bubbles to several sectors. Recently, econometric testing for rational bubbles has been examined intensively using different statistical approaches. We give as examples: testing on exploding dividends in US stock prices between 1974 and 2000 [46], an analysis of the S&P500 [48] and the G-7 stock markets [103]. Demos [32] discusses testing of bubble birth and bursting with many examples.

Here one has to mention **Harrison and Kreps** [52], because they were one of the first who formalised, still in a very basic setting, the principle of how bubbles work. Speculative behaviour means that an investor is willing to pay more than the expected, discounted value of future dividends, because he is aware that other investors are willing to do the same. In this case, the price of the asset can no longer be only justified by dividend payments, but is stable at some point. Harrison and Kreps [52] show the existence and uniqueness of such a balance. Without naming it an equilibrium price, this is exactly the idea behind it. Heterogeneity is the source of this gap between two prices, but they do not explain where these different beliefs come from. The model by Chen and Kohn [21], which we discuss later, reuses the ideas from [52]. Scheinkman and Xiong [90] provide a survey of the existing literature up to 2003 which summarises all the earlier models. We refer the reader to his work for the earlier models. Moreover, they introduce a more general equilibrium model [89, 90] for two different investor groups. The dividend rate is seen as a non-observable Ornstein-Uhlenbeck-process. Their provided model seems at first glance similar to our setting following [21]: Heterogeneous beliefs are the source of the bubble and an Ornstein-Uhlenbeck process models the dividend rate. However, they use different signals and linear filtering techniques to determine pairs of equilibrium prices. The market participants just observe the cumulated dividend, different "signals" for each group and different distorted "signals" correlated to the dividend rate. Using a linear filtering technique each investor group estimates the dividend rate. Under the assumption that a market price can be decomposed into a fundamental price and resale value, the resale value fulfils a hypergeometric differential equation and can be interpreted as a bubble. The model by Scheinkman and Xiong also allows to analyse bubbles in dependence of their parameters like transaction costs, interest rates or volatilities. A study of the boundary problem arising this setting and iterative algorithm [6,7] allows a generalisation of Scheinkman's idea to a time-dependent, dynamic framework.

1.5. Regime Switching

It is mainly the work of Robert J. Elliott that made regime switching part of modern financial mathematics. Nowadays, there is a large literature on this topic. The first book of Elliott [35] on *Hidden Markov Chains* provides the most important theoretical framework as well as methods of estimation. However, what we know as Markovian regime switching was developed later. The article *American Options with Regime Switching* [36] is a cornerstone for the theory. Elliott and Buffington consider a Black-Scholes economy in which all the parameters switch between a "good" and a "bad" state. In a straightforward way, they show that a Black-Scholes type equation can be obtained. The basic idea of regime switching is to combine classical models, such as a Black-Scholes environment [36, 37, 40], a LIBOR market model [38] or a Heath-Jarrow-Morton model [45] for example, with an additional stochastic component, the so called economic regime. On a filtered probability space $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ a model is a assumed, where at least one parameter changes over the time based on the economic regime, for example a piecewise constant interest rate. The regime changes according to a continuous-time Markov chain $X = (X_t)_{t\geq 0}$ with N different states. The state space is the set of unit vectors $\{\mathtt{e}_1, \dots, \mathtt{e}_N\}$ with rate matrix $\mathtt{A}$. Any parameter that depends on the regime, can therefore easily be written as an inner product, for instance, an interest rate

$$r_t = \langle \mathtt{r}, X_t \rangle \tag{1.11}$$

where $\mathtt{r}$ is a vector of different interest rates. This is called the canonic notation of the Markov states which facilitates all further calculation and is hence kept by the overwhelming part of authors. However, the biggest advantage is obviously, that more than one parameter can easily switch to another state at the same time, whenever the economic situation changes. The interpretation of X itself is vast and depends merely on the context of the model. It can be seen as an indicator that summarises all "information about some (macro)-economic factors such as Gross Domestic Product (GDP) and Retail Price Index (RPI)" as in [45] where the states of X can be interpreted as "different categories of credit ratings produced by rating agencies which are publicly available." The Markov chain, recalling the semi-martingale-representation [35], is in general decomposed as

$$X_t = X_0 + \int_0^t \mathtt{A} X_s ds + M_t \tag{1.12}$$

where M is a martingale with respect to the filtration generated by X. This is the most important tool to handle expectations containing a regime switching parameter. Due to the additional random component, the modified models become incomplete. We want to illustrate this within an example. In [37], the stock price is given by a Markov-modulated Geometric Brownian Motion. A straight forward measure change is not possible, since the appreciation rate and the volatility are stochastic. Moreover, let $\left(\mathcal{F}_t^X\right)_{t\geq 0}$ be the natural filtration generated by X and $\left(\mathcal{F}_t^Z\right)_{t\geq 0}$ generated by the stock's logarithmic return Z_t. The Esscher transformation

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} = \frac{\exp\left(\int_0^t \theta_s dZ_s\right)}{\mathbb{E}^{\mathbb{P}}\left(\exp\left(\int_0^t \theta_s dZ_s\right) \middle| \mathcal{F}_t^X\right)} \tag{1.13}$$

with θ determined by the model can be used to find an equivalent martingale measure. Elliott [37] justifies his choice for an Esscher transformation by a minimal entropy argument - an idea taken from Lévy models. Under a certain choice of θ, the measure $\mathbb{Q}$ is the minimal entropy martingale measure. This allows a pricing analogous to standard models.

The two books edited by Mamon and Elliott [38] and [39] contain important works on application of regime switching and estimation of parameters and are rounded up by numerical aspects. There has been an intense research on regime switching especially in a Black-Scholes-setting the last years: Pricing European options via a system of coupled Black-Scholes-like PDE [74], approximate pricing for barrier options [43], risk minimising portfolios [40] and discrete-time, inhomogeneous Markov chain approximation method to price options [47]. However, regime switching is not restricted to Black-Scholes nor a Brownian motion setting. The Dupire model was also considered under regime switching [44]. There, the compounded interest rate, the local volatility and the appreciation rate switch within a Markovian setting. As in [37], the risk-neutral Esscher measure is used for defining a price. Finally, they derive a formula for pricing European call options as a solution of a system of an initial value problem. In [42], Elliott et al. investigate the pricing of options in a jump-diffusion model with regime switching. Even in this case, they are able to stay in a standard setting. A generalised Esscher transformation is used to determine an equivalent martingale measure and a PIDE approach for the pricing. One of the main questions remains the choice of the risk neutral measure. [92] discusses three different approaches to choose equivalent martingale measures: a stochastic differential game, a general equilibrium approach and an Esscher transformation. For the first two, a dynamic programming principle is used to determine the martingale measure. The latest development [94] generalises to a non-Markovian regime switching.

1.6. Chen and Kohn's Model for Asset Bubbles

As we base the two following chapters on a model by **Chen and Kohn** [21], we give a short introduction to their model here. There is only one risky asset paying a dividend rate D_t at $t \geq 0$ modelled by an Ornstein-Uhlenbeck process $D = (D_t)_{t \geq 0}$ with two investor groups differing only in their assessment of the mean-reversion rate λ_i, but agreeing on the other parameters $\mu \in \mathbb{R}$ and $\sigma > 0$. This process D is given by the stochastic differential equation

$$dD_t = \lambda_i (\mu - D_t) dt + \sigma dW_t^i$$

where W denotes a Brownian motion. The definition of a bubble is standard: the difference between minimal equilibrium price and intrinsic value. These two basic concepts are crucial for understanding the model.

An intrinsic value is defined as the maximal expected amount of all future dividends. In Chen and Kohn's setting, this can be calculated explicitly by knowing that the solution of an Ornstein-Uhlenbeck equation.

Determining the equilibrium price is far more challenging. First, it can be shown that the minimal equilibrium price is unique. Then, Chen and Kohn set up a second order linear differential equation

$$\max (\lambda_1 (\mu - x), \lambda_2 (\mu - x)) \Phi'(x) + \frac{\sigma^2}{2} \Phi''(x) - r\Phi(x) + x = 0$$

that can be transformed into a Weber differential equation and, hence, can be solved. Its continuous solution with linear growth at infinity is unique. In a second step, this solution is identified as the minimum equilibrium price. With help of the Itô formula, Chen and Kohn [21] prove that such a solution is an equilibrium price. Then, through help of the theory of viscosity solutions, they show its minimality. An erratum [22] corrects some details of this proof. Finally, the size of the bubble - introduced as the difference between the minimal equilibrium price and the intrinsic value - can be computed explicitly.

The model is connected to other literature as mentioned above. It can be seen as a continuous generalisation of the Harrison and Kreps' [52] ideas. At first glance it also bears resemblance to the setting by Scheinkman and Xiong [89, 90], but the differences are significant. Despite also using Ornstein-Uhlenbeck processes for modelling the dividend rate, Scheinkman and Xiong assume that this process is not observable; the heterogeneity in opinion comes from different signal processes used for linear filtering. Moreover, Scheinkman and Xiong's concept of equilibrium prices is majorly different. Instead of expressing different opinions through different measures and connecting them by a measure change, they define pairs of equilibrium prices.

2. Preliminaries

2.1. An Introduction to Lévy Processes

In this chapter, we give an introduction to Lévy processes and discuss their basic properties. The notation used here follows mostly the book of **Sato** *Lévy Processes and Infinitely Divisible Distributions* [88]; therefore it should be noted that it differs slightly from the one used in the book of Applebaum [5]. A brief overview over the topic is also provided in the book by Cont and Tankov *Financial Modelling with Jump Processes* [24] as well as in Protter's book *Stochastic Integration and Differential Equations* [83].

Throughout this whole work, we assume $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ to be a filtered probability space satisfying the usual hypothesis. A stochastic process $L = (L_t)_{t\geq 0}$ is called **Lévy process** if the following conditions are satisfied:

- $L_0 = 0$ almost sure,
- L has *independent increments*, i.e. for $n \geq 1$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent.
- L has *stationary increments*, i.e. $L_t - L_s$ has the same distribution as L_{t-s} for $0 \leq s < t < \infty$,
- L is *continuous in probability*, i.e.

$$\lim_{s\to t} \mathbb{P}\left(|L_s - L_s| > \epsilon\right) = 0 \tag{2.1}$$

  for $t \geq 0$ and $\epsilon > 0$.

Like Sato [88], we additionally assume all Lévy processes to be *càdlàg*. However, this is not necessary and without this assumption, it can be shown that every Lévy process has a unique càdlàg modification that is itself a Lévy process (see Theorem 2.1.8 in [5] or Theorem 30 in [83]). Obviously, since a Lévy process is stationary and has independent increments, it is a Markov process. Famous examples for Lévy processes are the Brownian motion and the Poisson process. We denote the left limits of a process by

$$L_{t-} = \lim_{s\to t^-} L_s \tag{2.2}$$

and the jumps of a process by

$$\Delta L_t = L_t - L_{t-} \tag{2.3}$$

The big advantage of Lévy processes is that they allow to include those jumps into models while having a wide range of helpful theoretical results.

Lévy processes are closely related to infinitely divisible distributions (see Chapter 2 in [88]): If L is a Lévy process, then L_t has an infinitely divisible distribution for every $t \geq 0$. Conversely, for each infinitely divisible distribution F there exists a Lévy process L such that F is the distribution of L_1. An important result about infinitely divisible distributions, the **Lévy Khintchine formula**, can hence be also applied to Lévy processes. We receive

Theorem 1. *Let ν be a measure on $\mathbb{R}$ satisfying*

$$\int_{-\infty}^{\infty} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty. \tag{2.4}$$

Let $\gamma \in \mathbb{R}$, $a \geq 0$ and L be a Lévy process, then L has the characteristic function

$$\mathbb{E}\left(e^{iuL_t}\right) = \exp\left(t\left(i\gamma u - \frac{u^2a^2}{2} + \int_{\mathbb{R}\setminus 0}\left(e^{iux} - 1 - iu\mathbf{1}_{[-1,1]}(x)\right)\nu(dx)\right)\right) \tag{2.5}$$

for $u \in \mathbb{R}$.

The measure ν is called **Lévy measure** and (a, ν, γ) the generating or **characteristic triplet**. It can be further shown that the characteristic function of L_t is uniquely determined by this triplet and conversely, for each characteristic triplet there exists a Lévy process. A detailed proof can be found in Chapter 2.8 in [88]. For $A \in \mathcal{B}(\mathbb{R})$, the Lévy measure $\nu(A)$ can also be interpreted as the expected number of jumps with size belonging to A between zero and one; we can also write

$$\nu(A) = \mathbb{E}\left(\#\{t \in [0,1] : \Delta L_t \neq 0, \Delta L_t \in A\}\right). \tag{2.6}$$

With this, one can show that for a bounded, real-valued function f vanishing around zero holds

$$\mathbb{E}\left(\sum_{0<s\leq t} \Delta L_s\right) = t\int_{-\infty}^{\infty} f(x)\nu(dx). \tag{2.7}$$

As shown in Chapter 2 of [24], for every càdlàg process there is a random measure on $\mathbb{R} \times [0, \infty)$ describing the jumps. For a process L, we define its jump measure

$$J(B) = \#\{(t, L_t - L_{t-}) \in B\} \tag{2.8}$$

for $B \in \mathcal{B}(\mathbb{R} \times [0, \infty))$. Roughly spoken, $J(dz, dt)$ counts the number of jumps of L occurring in dt whose amplitude belong to dz. It is very useful as it helps us to represent all quantities involving the jumps of L through an integral against the jump measure J. We consider a compound Poisson process

$$X_t = \sum_{i=1}^{N_t} Y_i \tag{2.9}$$

with intensity λ and jump size distribution f. Then, its jump measure is a Poisson random measure with intensity $\lambda f(dx)dt = \nu(dx)dt$ (see Proposition 3.5 in [24]). One can show that every compound Poisson process has the representation

$$X_t = \sum_{s\in[0,t]} \Delta X_s = \int_0^t \int_{\mathbb{R}} zJ(dz, ds). \tag{2.10}$$

We could have the idea to restrict ourselves to processes of the form

$$L_t = \gamma t + W_t + X_t = \gamma t + aW_t + \int_0^t \int_{\mathbb{R}} zJ(dz, ds). \tag{2.11}$$

These are called **jump-diffusion processes**. In fact, jump-diffusion is quite commonly used in financial modelling, such as Merton's approach (see [25], and it is much easier to handle. Drift, Brownian motion and jumps are its basic components - as it is with Lévy processes, but not every Lévy process be represented in this form. A Lévy process may have an infinite number of small jumps. Considering this fact, we can formulate an important result.

Theorem 2 (Lévy Itô decomposition). *Let L be a Lévy process L with characteristic triplet (a, ν, γ) that satisfies*

$$\int_{|z|\leq 1} z\nu(dz) < \infty \tag{2.12}$$

and let $J(dz, ds)$ be its jump measure. Then the following holds:

- *$J(dz, ds)$ is a Poisson random measure on $[0, \infty)$ with intensity measure $\nu(dx)dt$.*
- *L has the **decomposition***

$$L_t = \gamma t + aW_t + X_t^1 + X_t^2 \tag{2.13}$$

where W is a standard Brownian motion,

$$X_t^1 = \int_0^t \int_{|z|>1} zJ(dz, ds) \tag{2.14}$$

describes the large jumps and

$$X_t^2 = \lim_{\varepsilon \to 0^-} \int_0^t \int_{\varepsilon<|z|\leq 1} z\left(J(dz, ds) - \nu(dz)ds\right) \tag{2.15}$$

the compensated small jumps.

- *The continuous part $\gamma t + aW_t$ and the jump part $X_t^1 + X_t^2$ are **independent**.*

For a more general version with a proof see Chapter 4, Theorem 19.2 and 19.3, in [88]. Obviously, the Lévy Khintchine formula can also be seen as a direct consequence of the Lévy Itô decomposition [24].

Lévy processes have the **Markov property** (see Chapter 3.8 in [24] or Chapter 3.1 in [5]). We say a process X has the Markov property if

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_t)|X_s) \tag{2.16}$$

for all $0 \leq s \leq t < \infty$ and all bounded and measurable functions f. In other words, the process after time s is independent on its past. The transition probability of a Markov process X is defined as

$$P_{s,t}(x, B) = \mathbb{P}\left(X_t \in B | X_s = x\right) \tag{2.17}$$

for all $B \in \mathcal{B}(\mathbb{R})$ and $0 \leq s \leq t < \infty$. It can be shown that for Lévy processes, they are homogeneous in space and time (Theorem 10.5 in [88]). The transition operator for Markov processes is defined as

$$T_{s,t}f(x) = \int_{\mathbb{R}} f(y)P_{s,t}(x, dx) \tag{2.18}$$

For homogeneous Markov processes, we write $T_{0,t}$ as T_t. Therefore, we get

$$P_t f(x) = \mathbb{E}(f(x + X_t)) \tag{2.19}$$

for $t \geq 0$. Using the Chapman-Kolmogorov identity (Theorem 3.1.5 from [5])

$$P_{s,u}(x, B) = \int_{\mathbb{R}} P_{s,u}(y, B)P_{s,t}(x, dy) \tag{2.20}$$

and the time homogeneity, one can show that the transition operators have the semigroup relation

$$P_t P_s = P_{s+t}. \tag{2.21}$$

The **infinitesimal generator** of a Lévy process is defined as

$$\mathrm{L}\, f = \lim_{t \downarrow 0} \frac{P_t f - f}{t} \tag{2.22}$$

on a domain such that the limit on the right hand side exists. An important result is

Theorem 3. *Let L be a Lévy process with generating triplet (a, ν, γ). Then, the infinitesimal generator can be written as*

$$\mathrm{L}\, f(x) = \frac{a}{2} f''(x) + \gamma f'(x) + \int_{-\infty}^{\infty} f(x+y) - f(x) - y f'(x) \mathbf{1}_{|x| \leq 1} \nu(dy) \tag{2.23}$$

For a multidimensional version with proof we refer to Chapter 6.31 in [88]. The infinitesimal generator appears often in partial integro-differential equations associated with Lévy models and can provide a useful tool. Consider a function $f \in C^2(\mathbb{R})$. We shall later see such a case in our model. For a Brownian motion, we receive $\mathrm{L}\, f = \frac{1}{2}\Delta$ with the Laplace operator Δ and for the compound Poisson process

$$\mathrm{L}\, f(x) = \int_{-\infty}^{\infty} f(x+y) - f(x) \nu(dy). \tag{2.24}$$

The following lemma is an important technical step we will need later. Proposition 11.10 in [88] shows that projections of Lévy processes are again Lévy processes as a consequence of the Lévy-Khinchin formula (Theorem 1).

Lemma 1. *Let L be a Lévy process with characteristic triplet (a, ν, γ) and $c \in \mathbb{R} \setminus \{0\}$. Then $(cL_t)_{t \geq 0}$ is a Lévy process with characteristic triplet (a_c, ν_c, γ_c) where*

$$\begin{aligned} a_c &= ca, \\ \gamma_c &= c\left(\gamma + \int_{-|c|}^{|c|} x \nu(dx) - \int_{-1}^{1} x \nu(dx)\right), \\ \nu_c(dx) &= \nu\left(\frac{dx}{c}\right). \end{aligned}$$

Lévy processes play an important rule in generalising famous stochastic differential equations. One of them is the Non-Gaussian Ornstein-Uhlenbeck process, also called **Lévy driven Ornstein-Uhlenbeck process**. Definitions are not unique in literature. As introduced in Chapter 15.3 in [24], one definition could be

$$Y_t = Y_0 e^{-rt} + \int_0^t e^{-r(t-s)} dL_s. \tag{2.25}$$

This allows us to find a stochastic differential equation and to calculate the characteristic triplet of the stationary distribution of Y_t. Estimation for Lévy driven Ornstein-Uhlenbeck processes can get complicated; in most cases, several additional assumptions are taken. For a detail analysis see Barndorff-Nielsen and Shepard [8].

However, we choose a slightly different definition. A very basic, but important result is

Lemma 2. *The stochastic differential Ornstein-Uhlenbeck equation*

$$dD_t = \lambda(\mu - D_{t-})dt + \sigma dL_t \tag{2.26}$$

has the unique solution

$$D_t = \mu + e^{-\lambda t}(x - \mu) + \sigma \int_0^t e^{-\lambda(t-s)} dL_s. \tag{2.27}$$

Proof. As a first step, we set $Y_t = \mu - D_t$ and observe $dY_t = -dD_t$. Note that Y_t has jumps, but is still a (discontinuous) semimartingale. Using the Itô-formula for general semimartingales (see [83, p. 78f]) onto $f(t, Y_t) = e^{\lambda_\iota t} Y_t$, we obtain

$$\begin{aligned} de^{\lambda t} Y_t &= \lambda e^{\lambda_\iota t} Y_{t-} dt + e^{\lambda t} dY_t + \sum_{0<s\leq t} \left(e^{\lambda s} Y_s - e^{\lambda s} Y_{s-} - e^{\lambda s} \Delta Y_s \right) \\ &= \lambda e^{\lambda_\iota t} Y_{t-} dt - \lambda_\iota e^{\lambda t} Y_{t-} dt - \sigma e^{\lambda t} dL_t = -\sigma e^{\lambda t} dL_t. \end{aligned}$$

By integration, we finally get

$$Y_t = e^{-\lambda_\iota t} \left(Y_0 - \sigma \int_0^t e^{\lambda_\iota s} dL_s^\iota \right) \tag{2.28}$$

and hence, the above given representation of D_t. □

Further interesting properties can be found in [5]. There is a closed form representation of the generator, see Example 6.7.6 in [5].

2.2. Stablility

In this Section, we discuss an important class of Lévy processes that share many properties like self-similarity with the Brownian motion: stable processes. For a detailed discussion see Samorodnitsky and Taqqu's book *Stable Non-Gaussian Random Processes* [86] or Chapter 3 in Sato [88]. Section 3.7 in [24] is rather comprehensive and focussed on the most important properties.

There are several equivalent definitions of stability (see Chapter 1 in [86]). We call a distribution of a random vector X stable if for any $A, B > 0$, there exists $C > 0$ and $D \in \mathbb{R}$ such that

$$AX_1 + BX_2 \stackrel{d}{=} EX + D \tag{2.29}$$

where X_1 and X_2 are independent copies of X and $\stackrel{d}{=}$ means equality in distribution. A Lévy process is called **stable** if the distribution of L_1 is stable, or alternatively, when for every $a > 0$ there exist a $b(a) > 0$ and $c(a) > 0$ such that

$$\mathbb{E}\left(e^{iuL_1}\right)^a = \mathbb{E}\left(e^{ib(a)uL_1}\right)e^{ic(a)u} \tag{2.30}$$

and strictly stable if

$$\mathbb{E}\left(e^{iuL_1}\right)^a = \mathbb{E}\left(e^{ib(a)uL_1}\right). \tag{2.31}$$

One can show that for every strictly stable process, there exists an $\alpha \in (0, 2]$ such that $b(a) = a^{1/\alpha}$ (see Corollary 2.1.3 from [86]); this α is called index of stability. Stable processes with index α are also known as **α-stable processes**. For such processes there is a $c > 0$ such that

$$\left(a^{-\frac{1}{\alpha}} L_{at} + tc\right)_{t \geq 0} \stackrel{d}{=} (L_t)_{t \geq 0} \tag{2.32}$$

for all $a > 0$. If $c = 0$, the process is obviously self-similar. For our purposes, we simplify Theorem 14.3 from [88], an important result about stable distributions (see also Proposition 3.15 in [24]), to

Lemma 3. *A non-trivial, infinitely divisible distribution with the characteristic triplet (a, ν, γ) is α-stable with $0 < \alpha < 2$ if and only if $\gamma = 0$ and there exists a finite measure λ such that*

$$\nu(B) = \int_0^1 \lambda(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) \frac{dr}{r^{1+\alpha}} \tag{2.33}$$

for $B \in \mathcal{B}(\mathbb{R})$. A distribution is α-stable with $\alpha = 2$ if and only if it is Gaussian.

The function $r^{-\alpha-1}$ is obviously increasing in α for $0 < r < 1$ and decreasing for $1 < r < \infty$. Hence, α-stable processes move mainly by big jumps as α is close to 0 and mainly by small jumps if α is near 2. Rewriting Lemma 3, it follows that for an α-stable Lévy process with $a = 0$ and $0 < \alpha < 2$ holds

$$\nu(dx) = c_1 \frac{\mathbf{1}_{(0,\infty)}(x)}{x^{\alpha+1}} dx + c_2 \frac{\mathbf{1}_{(-\infty,0)}(x)}{|x|^{\alpha+1}} dx \tag{2.34}$$

where $c_1 \geq 0$, $c_2 \geq 0$ and $c_1 + c_2 \geq 0$. We will use this notation throughout all further chapters. The only 2-stable processes are Gaussian (Theorem 14.1 in [88]); as they don't have jumps, $\nu = 0$. With some effort, it is possible to find a closed form representation for the characteristic function. Theorem 14.15 from [88] gives us

Theorem 4. *If L is a non-trivial α-stable process with $0 < \alpha < 2$, the*

$$\mathbb{E}\left(e^{iuL_1}\right) = \begin{cases} \exp\left(imu - |u|^\alpha \sigma^\alpha \left(1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right) & \text{for } \alpha \neq 1, \\ \exp\left(imu - |u|\sigma \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log\left(|u|\right)\right)\right) & \text{for } \alpha = 1 \end{cases} \tag{2.35}$$

with $\sigma > 0$, $-1 \leq \beta \leq 1$ and $m \in \mathbb{R}$. These constants are uniquely determined. Conversely, for every $\sigma > 0$, $-1 \leq \beta \leq 1$ and $m \in \mathbb{R}$ there exists a non-trivial α-stable process with $0 < \alpha < 2$ satisfying (2.35).

Hence, four parameters uniquely determine a stable process and one also finds the term **stable process with parameters** $(\alpha, \beta, m, \sigma)$. This definition (from [88]), unfortunately, is just one of many slightly different notations in the literature and one should be very careful. If the parameter β is zero, the process is symmetric; if $\beta = 1$ the process is on $(0, \infty)$ and if $\beta = -1$, the process is on $(-\infty, 0)$.

There are three important examples where a closed form of the density is known:

- $(2, \beta, m, \sigma)$: the **Gaussian process** with $L_1 \sim N(m, \sigma^2)$,
- $(1, 0, m, \sigma)$: the **Cauchy process** with density

$$f_{L_1}(x) = \frac{\sigma}{\pi((x-m)^2 + \sigma^2)}, \tag{2.36}$$

- $(\frac{1}{2}, 1, m, \sigma)$: the **process having Lévy distribution** with density

$$f_{L_1}(x) = \sqrt{\frac{\sigma}{2\pi\,(x-m)^3}} \exp\left(-\frac{\sigma}{2(x-m)}\right) \tag{2.37}$$

  for $x > m$.

However, there is a power series representation of the density. Among other examples without a closed form density, we want to particularly mention the **Meixner process**; an Ornstein-Uhlenbeck process driven by a Meixner process is studied intensively in [50].

In the last part of this section, we prove a theoretical result about stable processes, we shall later need. In order to avoid confusion, we define a **Lévy process with α-stable jumps** as a Lévy process such that its jump part (in the sense of the Lévy Itô decomposition) is α-stable. From Lemma 1, we already know that linear transformations of Lévy processes are again Lévy processes.

Lemma 4. *Let L be a Lévy process with characteristic triplet (a, ν, γ) and α-stable jumps. Let $\sigma > 0$. Then the process $(\sigma L_t)_{t \geq 0}$ is also a Lévy process with characteristic triplet $(\breve{a}, \breve{\nu}, \breve{\gamma})$ where*

$$\begin{aligned} \breve{a} &= \sigma a, \\ \breve{\nu}(dx) &= \sigma^\alpha \nu(dx), \\ \breve{\gamma} &= \begin{cases} \sigma\left(\gamma - \frac{\sigma^{\alpha-1}-1}{\alpha-1}(c_2 - c_1)\right) & \text{if } \alpha \neq 1, \\ \sigma\left(\gamma - \log(\sigma)(c_2 - c_1)\right) & \text{if } \alpha = 1 \end{cases} \end{aligned}$$

and the jump part is α-stable.

Proof. The process $(\sigma L_t)_{t\geq 0}$ is obviously a Lévy process. Making use of Theorem 1, we determine the characteristic triplet. For $\alpha \neq 1$, we examine

$$\begin{aligned}
&\int_{\mathbb{R}\setminus 0} \left(e^{i\sigma ux} - 1 - i\sigma ux\mathbf{1}_{|x|\leq 1}\right) \nu(dx) \\
&\qquad = \int_{\mathbb{R}\setminus 0} \left(e^{iuy} - 1 - iuy\mathbf{1}_{|y|\leq \sigma}\right) \left(\frac{c_1\mathbf{1}_{y<0}}{y^{\alpha+1}} + \frac{c_2\mathbf{1}_{y>0}}{|y|^{\alpha+1}}\right) \sigma^{\alpha+1}\frac{dy}{\sigma} \\
&\qquad = \int_{\mathbb{R}\setminus 0} \left(e^{iuy} - 1 - iuy\mathbf{1}_{|y|\leq \sigma}\right) \sigma^{\alpha}\nu(dy) \\
&\qquad = \int_{\mathbb{R}\setminus 0} \left(e^{iuy} - 1 - iuy\mathbf{1}_{|y|\leq 1}\right) \sigma^{\alpha}\nu(dy) + iu\frac{\sigma^{\alpha} - \sigma}{\alpha - 1}(c_2 - c_1).
\end{aligned}$$

The case $\alpha = 1$ is similar. Hence, we obtain

$$\begin{aligned}
&\mathbb{E}\left(e^{iu(\sigma L_1)}\right) = \mathbb{E}\left(e^{i(\sigma u)L_1}\right) \\
&\quad = \exp\left(i\gamma(\sigma u) - \frac{(\sigma u)^2 a^2}{2} + \int_{\mathbb{R}\setminus 0} \left(e^{i(\sigma u)x} - 1 - i(\sigma u)x\mathbf{1}_{|x|\leq 1}\right) \nu(dx)\right) \\
&\quad = \exp\left(i\breve{\gamma}u - \frac{\breve{a}^2 u^2}{2} + \int_{\mathbb{R}\setminus 0} \left(e^{iuy} - 1 - iuy\mathbf{1}_{|y|\leq 1}\right) \breve{\nu}(dy)\right).
\end{aligned}$$

Since we assumed $\sigma > 0$, the process is also α-stable. □

2.3. Change of Measure

Changing the measure is an important step in many parts of this work. Therefore, we need the Girsanov theorem (see Theorem 39 in [83] or [68, p. 190ff] for a proof). In its general version it can be formulated as following

Theorem 5 (Grisanov theorem for semimartingales). *Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent measures. Let X be a semimartingale under $\mathbb{P}$ with the Doob-Meyer decomposition $X = M + A$. Then X is also a semimartingale under $\mathbb{Q}$ and has a decomposition $X = L + C$, where*

$$L_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s \tag{2.38}$$

is a $\mathbb{Q}$ local martingale, and $C = X - L$ is a $\mathbb{Q}$ finite variation process.

A very useful and important tool for verifying the martingale property is the Novikov condition (see Theorem 45 in [83]). We have

Theorem 6 (Novikov condition). *For a continuous local martingale satisfying*

$$\mathbb{E}\left(\exp\left(\frac{1}{2}[M, M]_\infty\right)\right) < \infty, \tag{2.39}$$

the stochastic exponential $\mathcal{E}(M)$ is a uniformly integrable martingale.

A special version of the Girsanov theorem (see Theorem 46 in [83]) is often used. From Theorem 5 and Lévy's theorem (see Theorem 39 in [83]) one can show

Theorem 7 (Girsanov theorem for Brownian motion). *Let W be a Brownian motion under $\mathbb{P}$ and H adapted, càglàd and bounded. Define B by*

$$B_t = \int_0^t H_s ds + W_t \tag{2.40}$$

and define $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T H_s dW_s - \int_0^T H_s^2 ds\right) \tag{2.41}$$

for some $T > 0$. Then B is Brownian motion under $\mathbb{Q}$ for $0 \leq t \leq T$.

It is of importance to notice that there can be shown a version of Girsanov's measure change for Lévy processes, see Theorems 33.1 and 33.2 in [88] or Chapter 9.4 in [24]. [24] gives the example for tempered α-stable processes, but for α-stable distributions our case will violate the assumptions.

2.4. Pseudo-Differential Operators

Apart from their application in the theory of partial differential equations, pseudo-differential operators are closely related to infinitesimal generators of Lévy processes (see Chapter 6.31 in [88]). Following the book of Wong [102], we summarise the theory of pseudo-differential operators by giving some basic definitions and results and to introduce the concept of weak and strong solutions. Details, proofs and examples can be found in [102].

On the space $\mathbb{R}^n$, we define the differential operator

$$\mathsf{D}_j = -\frac{\partial}{\partial x_j} \tag{2.42}$$

for $j \in \{1, \ldots, n\}$. Using a multi-index $\boldsymbol{\alpha} \in \mathbb{N}^n$, the operator

$$P(x, \mathsf{D}) = \sum_{|\boldsymbol{\alpha}| \leq m} a_{\boldsymbol{\alpha}}(x) \mathsf{D}^{\boldsymbol{\alpha}} \tag{2.43}$$

is a linear partial differential operator of order $m \in \mathbb{N}$. Then, we can define the **Schwartz space** or space of rapidly decreasing functions as

$$\mathcal{S} = \left\{ \varphi \in C_0^\infty (\mathbb{R}^n) \ : \ \sup_{x \in \mathbb{R}^n} \left| x^{\boldsymbol{\alpha}} \mathsf{D}^{\boldsymbol{\beta}} \varphi(x) \right| < \infty \ \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n \right\}. \tag{2.44}$$

In other words, this describes the set of all infinitely differentiable functions whose partial derivatives all decrease faster than any power of $x^{-\boldsymbol{\alpha}}$. The inclusion of $\mathcal{S}$ in $C_0^\infty (\mathbb{R}^n)$ is proper, since $e^{-x^2} \in \mathcal{S}$, but obviously not in $C_0^\infty (\mathbb{R}^n)$. Moreover, the Schwartz space $\mathcal{S}$ is dense in $L^p (\mathbb{R}^n)$ for $1 \leq p \leq \infty$ (see Theorem 3.9 in [102]). Further, we define a set

$$S^m = \{ \varsigma(x, \xi) \in C_0^\infty (\mathbb{R}^n \times \mathbb{R}^n) \ : \ \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n \ \exists C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0 \ : \\ \left| \mathsf{D}_x^{\boldsymbol{\alpha}} \mathsf{D}_\xi^{\boldsymbol{\beta}} \varsigma(x, \xi) \right| \leq C_{\boldsymbol{\alpha},\boldsymbol{\beta}} (1 - |\xi|)^{m - |\beta|} \ \forall x, \xi \in \mathbb{R}^n \Big\} \tag{2.45}$$

for any $m \in \mathbb{R}$. Let us consider function $\varsigma \in \cup_{m \in \mathbb{R}} S^m$. With

$$\hat{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \varphi(x) dx \tag{2.46}$$

we denote the Fourier transformed of $\varphi \in \mathcal{S}$. Then, for a function $\varsigma : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, we define an operator T_ς as **pseudo-differential operator**, if

$$T_\varsigma \varphi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \varsigma(x, \xi) \hat{\varphi}(\xi) d\xi \tag{2.47}$$

for $\varphi \in \mathcal{S}$ holds. The function ς is called its **symbol**. Every linear partial differential operator $P(x, \mathsf{D})$ with $a_{\boldsymbol{\alpha}}(x) \in C^\infty$ is also a pseudo-differential operator. By the properties of the Fourier transformation, one can represent (see Chapter 6 in [102]) a partial differential operator as

$$P(x, \mathsf{D}) = \sum_{|\boldsymbol{\alpha}| \leq m} a_{\boldsymbol{\alpha}}(x) \mathsf{D}^{\boldsymbol{\alpha}} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} P(x, \xi) \hat{\varphi}(\xi) d\xi. \tag{2.48}$$

Hence it is easy to show that $P(x, \xi) = \sum_{|\alpha| \leq m} a_{\boldsymbol{\alpha}}(x) \xi^{\boldsymbol{\alpha}}$ is in this case its symbol and a polynomial in ξ. A pseudo-differential operator maps the Schwartz space into itself. The composition of two pseudo-differential operators is again pseudo-differential operator (see Chapter 8 in [102]). For any pair $\varphi, \psi \in \mathcal{S}$ we introduce an inner product as

$$(\varphi, \psi) = \int_{\mathbb{R}^n} \varphi(x)\overline{\psi(x)}dx. \tag{2.49}$$

It can be shown that there exists a formal **adjoint** operator $T_\varsigma^* : \mathcal{S} \to \mathcal{S}$, i.e.

$$(T_\varsigma \varphi, \psi) = \left(\varphi, T_\varsigma^* \psi\right) \tag{2.50}$$

for every $\varphi, \psi \in \mathcal{S}$. ref It is again a pseudo-differential operator. Further, T_ς and its symbol $\varsigma \in S^m$ are called **elliptic** if there exist $C, R > 0$ such that

$$|\varsigma(x,\xi)| \geq C\left(1 - |\xi|\right)^m \tag{2.51}$$

for $|\xi| \geq R$. In Chapter 10 of Wong [102], we find an approximation to the inverse of elliptic operators. If there are $C, R > 0$, such that the inequality

$$\operatorname{Re} \varsigma(x,\xi) \geq C(1 + |\xi|)^m \tag{2.52}$$

holds for all $|\xi| \geq R$, then the operator T_ς is said to be **strongly elliptic**. In order to introduce the solution concepts for pseudo-differential equations, we have to repeat some definitions and properties from functional analysis. For a linear operator $T : X \to Y$, a dense subspace of X is denoted as $\mathcal{D}(T)$ and called the **domain** of T. An operator is said to be **closed** if for any sequence $(x_k)_{k\geq 0}$ in $\mathcal{D}(T)$ such that $x_k \to x$ in X and $Tx_k \to y$ in Y as $k \to \infty$, we have $x \in \mathcal{D}(T)$ and $Tx = y$. For two operators T_1 and T_2, with domains $\mathcal{D}(T_1)$ and $\mathcal{D}(T_2)$ respectively, T_2 is called **extension** of T_1 if $\mathcal{D}(T_1) \subseteq \mathcal{D}(T_2)$ and $T_1 x = T_2 x$ for all $x \in \mathcal{D}(T_1)$. With $T_{\varsigma,0}$ we denote the smallest closed extension of T_ς called the **minimal operator**. We remark that the domain $\mathcal{D}(T_{\varsigma,0})$ consists of all functions $u \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$ for which a sequence $(\varphi_k)_{k\geq 0}$ in $\mathcal{S}$ exists such that $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\varsigma \varphi_k \to f$ in $L^p(\mathbb{R}^n)$ for some $f \in L^p(\mathbb{R}^n)$ as $k \to \infty$. Now let $u, f \in L^p(\mathbb{R}^n)$. We say $u \in \mathcal{D}(T_{\varsigma,1})$ and $T_{\varsigma,1} u = f$ if and only if

$$\left(u, T_\varsigma^* \varphi\right) = (f, \varphi) \tag{2.53}$$

for $\varphi \in \mathcal{S}$. One can show that $T_{\varsigma,1}$ is the largest closed extension of T_ς and hence, called the **maximal operator**. If the symbol $\varsigma \in S^m$ is elliptic, then $T_{\varsigma,0} = T_{\varsigma,1}$. Let us examine the **pseudo-differential equation**

$$T_\varsigma u = f \tag{2.54}$$

where $u, f \in L^p(\mathbb{R}^n)$. Let us restrict to symbols $\varsigma \in S^m$ with $m > 0$. Then, the function u is said to be a **weak solution** of $T_\varsigma u = f$ on $\mathbb{R}^n$ if

$$\left(u, T_\varsigma^* \varphi\right) = (f, \varphi) \tag{2.55}$$

for every $\varphi \in \mathcal{S}$. This is equivalent to the fact that $u \in \mathcal{D}(T_{\varsigma,1})$ and $T_{\varsigma,1} u = f$. Now, we characterise the functions f for which the pseudo-differential equation has a solution.

Lemma 5. *The pseudo-differential equation $T_\varsigma u = f$ on $\mathbb{R}^n$ has a weak solution u in $L^p(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that*

$$|(f, \varphi)| \leq C \left\| T_\varsigma^* \varphi \right\|_{p'} \tag{2.56}$$

for every $\varphi \in \mathcal{S}$, where p' and p are conjugate.

A function $u \in L^p(\mathbb{R}^n)$ that satisfies $T_{\varsigma,0}u = f$ and $u \in \mathcal{D}(T_{\varsigma,0})$ is called **strong solution** of $T_\varsigma u = f$. Obviously, strong solutions are also weak solutions. If ς is elliptic, then all weak solutions are also strong solutions. The following two important theorems give conditions for the existence and uniqueness of strong solutions.

Theorem 8. *Let $\varsigma \in S^m$ be a strongly elliptic symbol. Let I denote the identity operator. Then there exists $\lambda_0 \in \mathbb{R}$ such that for all $f \in L^2(\mathbb{R}^n)$ and $\lambda \geq \lambda_0$, the pseudo-differential equation $(T_\varsigma + \lambda I)\, u = f$ on $\mathbb{R}^n$ has a unique strong solution in $L^2(\mathbb{R}^n)$.*

Theorem 9 (Existence of a strong solution. Theorem 18.6 from [102])**.** *Let $\varsigma \in S^m$ be an elliptic symbol independent of x and $\varsigma(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. Then for every function $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, the pseudo-differential equation $T_\varsigma u = f$ on $\mathbb{R}^n$ has a unique strong solution in in $L^p(\mathbb{R}^n)$.*

2.5. Viscosity Solutions for PIDE

The theory of viscosity solutions provide us with a definition concept for a solution without imposing the existence of derivatives in advance. Introduced Crandall [30], it has been generalised to partial integro differential equations (PIDE) under additional assumptions (see [3, 24] or [25]). We only give the basic definitions and the notation in a simplified version from chapter 12.2.4 from [24].

A locally bounded function $u : \mathbb{R} \to \mathbb{R}$ is called upper-semicontinuous if $x_k \to x$ implies

$$u(x) \geq \limsup_{k\to\infty} u(x_k) = \lim_{k_0\to\infty} \sup_{k\geq k_0} u(x_k) \tag{2.57}$$

and **lower-semicontinuous** if $x_k \to x$ implies

$$u(x) \leq \liminf_{k\to\infty} u(x_k) = \lim_{k_0\to\infty} \inf_{k\geq k_0} u(x_k) \tag{2.58}$$

A function u is defined as continuous if it is upper- and lower-semicontinuous. The set $C_p^+(\mathbb{R})$ is defined as the of measurable functions on $\mathbb{R}$ with polynomial growth of degree p at positive infinity and bounded on $\mathbb{R}^-$ such that

$$\varphi \in C_p^+([0,T] \times \mathbb{R} \Leftrightarrow \exists C > 0 : |\varphi(x)| \leq C\left(1 + |x|^p \mathbf{1}_{x>0}\right). \tag{2.59}$$

To simplify the notation, we restrict ourselves to partial integro differential operators on $C^2(\mathbb{R}) \cap C_p^+(\mathbb{R})$ of the form

$$\begin{aligned} Lu(x) = a_1(x)u''(x) + a_2(x)u'(x) + a_3(x)u(x) + f(x) \\ + \int_{-\infty}^{\infty} u(x+y) - u(x) - y\frac{u(x)}{\partial x}\mathbf{1}_{|x|\leq 1}\nu(dy) \end{aligned} \tag{2.60}$$

where $a_1(x), a_2(x), a_3(x)$ are real valued functions. The polynomial growth condition makes it well-defined (see [25]). Then, we are able to define viscosity solutions in a still rather simple way (compare it with Definition 12.1 in [24]). A function u is a viscosity subsolution of $Lu = 0$ if for any real x and any test function $\varphi \in C^2(\mathbb{R}) \cap C_p^+(\mathbb{R})$ such that $u - \varphi$ has a global maximum point at $\hat{x}$, the following holds

$$\begin{aligned} a_1(\hat{x})\varphi''(\hat{x}) + a_2(\hat{x})\varphi'(\hat{x}) + a_3(\hat{x})u(\hat{x}) + f(\hat{x}) \\ + \int_{-\infty}^{\infty} \varphi(\hat{x}+y) - \varphi(\hat{x}) - y\varphi'(\hat{x})\mathbf{1}_{|\hat{x}|\leq 1}\nu(dy) \leq 0. \end{aligned} \tag{2.61}$$

A function u is a viscosity supersolution of $Lu = 0$ if for any real x and any test function $\varphi \in C^2(\mathbb{R}) \cap C_p^+(\mathbb{R})$ such that $u - \varphi$ has a global minimum point at $\hat{x}$, the following holds

$$\begin{aligned} a_1(\hat{x})\varphi''(\hat{x}) + a_2(\hat{x})\varphi'(\hat{x}) + a_3(\hat{x})u(\hat{x}) + f(\hat{x}) \\ + \int_{-\infty}^{\infty} \varphi(\hat{x}+y) - \varphi(\hat{x}) - y\varphi'(\hat{x})\mathbf{1}_{|\hat{x}|\leq 1}\nu(dy) \geq 0. \end{aligned} \tag{2.62}$$

A function is called a **viscosity solution** if it is a subsolution and a supersolution.

3. A Square Root Diffusion Version of Chen and Kohn's Model

The question is how sensitive is the model of Chen and Kohn [21] to changes in the setting. This Chapter will help us understanding the mechanisms behind the model better. In the first step, we replace the Ornstein-Uhlenbeck by a square root diffusion process

$$dD_t = \lambda_i \left(\mu - D_t\right) dt + \sigma \sqrt{D_t} dW_t^i \tag{3.1}$$

to model the dividend rate. With the restriction $2\lambda_2\mu \geq \sigma^2$ the square root diffusion process stays almost surely positive. Let $\left(W_t^1\right)_{t\geq 0}$ denote a Brownian motion on $\left(\Omega, \mathcal{A}, P^1\right)$. By the Girsanov theorem (see Theorem 7), the process $\left(W_t^2\right)_{t\geq 0}$ defined by

$$dW_t^2 = dW_t^1 + \frac{\left(\lambda_1 - \lambda_2\right)\left(\mu - D_t\right)}{\sigma\sqrt{D_t}} dt \tag{3.2}$$

is a Brownian motion on the probability space $\left(\Omega, \mathcal{A}, P^2\right)$, where P^2 and P^1 are equivalent measures. Trading, intrinsic values and equilibria can be introduced exactly as in in Chen and Kohn's [21] model. Using the fact

$$\left.\frac{e^{\lambda_i(s-t)} D_s}{\tau(s-t)}\right| \mathcal{F}_t \sim \chi^2_{\left(\frac{4\lambda_i\mu}{\sigma^2}\right)} \left(\frac{D_t}{\tau(s-t)}\right) \tag{3.3}$$

with $\tau(x) = \frac{\sigma^2}{4\lambda_i}\left(e^{\lambda_i x} - 1\right)$ for $t < s$ we can determine the conditional expectation

$$\begin{aligned}
\mathbb{E}^{P^i}\left(D_s | D_t = x\right) &= \frac{\tau(s-t)}{e^{\lambda_i(s-t)}} \left(\frac{4\lambda_i\mu}{\sigma^2} + \frac{x}{\tau(s-t)}\right) \\
&= e^{-\lambda_i(s-t)} \left(\mu\left(e^{\lambda_i(s-t)} - 1\right) + x\right).
\end{aligned}$$

Thus, the intrinsic value can be written as

$$I(x) = I\left(x, t\right) = \begin{cases} \frac{x}{r+\lambda_1} + \frac{\mu\lambda_1}{r(r+\lambda_1)} & \text{for } x < \mu, \\ \frac{x}{r+\lambda_2} + \frac{\mu\lambda_2}{r(r+\lambda_2)} & \text{for } x \geq \mu \end{cases} \tag{3.4}$$

which is remarkable, because it is the same as in the Ornstein-Uhlenbeck case. The existence and time independence of the minimal equilibrium price is shown exactly as in [21]. However, the equilibrium price is not the same. We need to examine another differential equation which is due to the structure of the model still transformable into a Kummer equation.

Lemma 6. *The differential equation*

$$\frac{\sigma^2}{2} x\Phi''(x) + \lambda_i \left(\mu - x\right) \Phi'(x) - r\Phi(x) + x = 0 \tag{3.5}$$

has a the general solution

$$\Phi(x) = c_1 \,\mathrm{M}\left(\frac{r}{\lambda_i}, \frac{2\lambda_i\mu}{\sigma^2}, \frac{2\lambda_i}{\sigma^2}x\right) + c_2 \,\mathrm{U}\left(\frac{r}{\lambda_i}, \frac{2\lambda_i\mu}{\sigma^2}, \frac{2\lambda_i}{\sigma^2}x\right) + \frac{x}{r+\lambda_i} + \frac{\mu\lambda_i}{r(r+\lambda_i)} \tag{3.6}$$

with real constants c_1 and c_2. The Kummer functions are defined (see [1]) as

$$\begin{aligned}
\mathrm{M}(a,b,x) &= \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\, k!} x^k, \\
\mathrm{U}(a,b,x) &= \frac{\pi}{\sin(\pi b)} \left(\frac{\mathrm{M}(a,b,x)}{\Gamma(1+a-b)\Gamma(b)} - x^{1-b} \frac{\mathrm{M}(1+a-b,2-b,x)}{\Gamma(a)\Gamma(2-b)} \right),
\end{aligned}$$

for $x \in \mathbb{R}$.

Proof. First, we transform the equation with

$$\Phi(x) = \Psi(x) + \frac{x}{r+\lambda_i} + \frac{\mu\lambda_i}{r(r+\lambda_i)} \tag{3.7}$$

into

$$\frac{\sigma^2}{2\lambda_i} x\Psi''(x) + (\mu - x)\,\Psi'(x) - \frac{r}{\lambda_i}\Psi(x) = 0. \tag{3.8}$$

Setting

$$\tilde{x} = \frac{2\lambda_i}{\sigma^2} x, \tag{3.9}$$

we get a confluent hypergeometric differential equation

$$\tilde{x} \frac{d^2}{d\tilde{x}^2} \Psi(\tilde{x}) + \left(\frac{2\lambda_i \mu}{\sigma^2} - \tilde{x} \right) \frac{d}{d\tilde{x}} \Psi(\tilde{x}) - \frac{r}{\lambda_i} \Psi(\tilde{x}) = 0, \tag{3.10}$$

which has the two given linearly independent solutions (see [1, p. 504ff]). □

Lemma 7. *The differential equation*

$$\max\left(\lambda_1(\mu - x), \lambda_2(\mu - x)\right) \Phi'(x) + \frac{\sigma^2}{2} x\Phi''(x) - r\Phi(x) + x = 0 \tag{3.11}$$

has a continuously differentiable solution with $\Phi(x) = O(x)$ for $x \to \infty$. This solution can be written as

$$\Phi(x) = \begin{cases} \mathrm{M}\left(\frac{r}{\lambda_1}, \frac{2\lambda_1\mu}{\sigma^2}, \frac{2\lambda_1}{\sigma^2}x\right) C_1 + \frac{x}{r+\lambda_1} + \frac{\mu\lambda_1}{r(r+\lambda_1)} & \text{for } x < \mu, \\ \mathrm{U}\left(\frac{r}{\lambda_2}, \frac{2\lambda_2\mu}{\sigma^2}, \frac{2\lambda_2}{\sigma^2}x\right) C_2 + \frac{x}{r+\lambda_2} + \frac{\mu\lambda_2}{r(r+\lambda_2)} & \text{for } x \geq \mu. \end{cases} \tag{3.12}$$

The constants are

$$\begin{aligned}
C_1 &= \frac{\xi_2\lambda_1(\lambda_1 - \lambda_2)\mu\sigma^2}{r(\lambda_1 + r)(\lambda_2 + r)(2\xi_1\xi_4\lambda_1\mu + \xi_2\xi_3\sigma^2)}, \\
C_2 &= \frac{\xi_1\lambda_1(\lambda_1 - \lambda_2)\mu\sigma^2}{r(\lambda_1 + r)(\lambda_2 + r)(2\xi_1\xi_4\lambda_1\mu + \xi_2\xi_3\sigma^2)}
\end{aligned}$$

with

$$\begin{aligned}
\xi_1 &= \mathrm{M}\left(\frac{r}{\lambda_1}, \frac{2\lambda_1\mu}{\sigma^2}, \frac{2\lambda_1\mu}{\sigma^2}\right), \\
\xi_2 &= \mathrm{U}\left(\frac{r}{\lambda_2}, \frac{2\lambda_2\mu}{\sigma^2}, \frac{2\lambda_2\mu}{\sigma^2}\right), \\
\xi_3 &= \mathrm{M}\left(\frac{\lambda_1 + r}{\lambda_1}, \frac{2\lambda_1\mu + \sigma^2}{\sigma^2}, \frac{2\lambda_1\mu}{\sigma^2}\right), \\
\xi_4 &= \mathrm{U}\left(\frac{\lambda_2 + r}{\lambda_2}, \frac{2\lambda_2\mu + \sigma^2}{\sigma^2}, \frac{2\lambda_2\mu}{\sigma^2}\right).
\end{aligned}$$

Proof. From Lemma 6 we get the general solution

$$\Phi(x) = c_1 \,\mathrm{M}\left(\frac{r}{\lambda_1}, \frac{2\lambda_1\mu}{\sigma^2}, \frac{2\lambda_1}{\sigma^2}x\right) + c_2 \,\mathrm{U}\left(\frac{r}{\lambda_1}, \frac{2\lambda_1\mu}{\sigma^2}, \frac{2\lambda_1}{\sigma^2}x\right) + \frac{x}{r+\lambda_1} + \frac{\mu\lambda_1}{r(r+\lambda_1)} \quad (3.13)$$

for $x < \mu$ and

$$\Phi(x) = c_3 \,\mathrm{M}\left(\frac{r}{\lambda_2}, \frac{2\lambda_2\mu}{\sigma^2}, \frac{2\lambda_2}{\sigma^2}x\right) + c_4 \,\mathrm{U}\left(\frac{r}{\lambda_2}, \frac{2\lambda_2\mu}{\sigma^2}, \frac{2\lambda_2}{\sigma^2}x\right) + \frac{x}{r+\lambda_2} + \frac{\mu\lambda_2}{r(r+\lambda_2)} \quad (3.14)$$

for $x \geq \mu$. The solution has a singularity in $x = 0$, because the Kummer U is not defined in this point. Under the condition $2\lambda_2\mu \geq \sigma^2$, the inequality $\frac{2\lambda_1\mu}{\sigma^2} > 1$ always holds. Therefore, it can easily be seen from the definition of the Kummer functions, that the limit

$$\lim_{x\to 0^+} \mathrm{U}\left(\frac{r}{\lambda_1}, \frac{2\lambda_1\mu}{\sigma^2}, \frac{2\lambda_1}{\sigma^2}x\right) \quad (3.15)$$

cannot exist. So we set the constant $c_2 = 0$. We recall the asymptotic behaviour of the Kummer functions (see [1, p. 504]) for $x \to \infty$,

$$\begin{aligned} \mathrm{M}(a,b,x) &= \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}\left(1 + O\left(x^{-1}\right)\right), \\ \mathrm{U}(a,b,x) &= x^{-a}\left(1 + O\left(x^{-1}\right)\right). \end{aligned}$$

In order to get $\Phi(x) = O(x)$, the constant c_3 obviously must be zero. As we want continuous differentiability of $\Phi(x)$, we must choose the remaining constant such that the function and its derivative are continuous in $x = \mu$. Hence, we obtain a system of linear equations in c_1 and c_4. Solving this system and renaming the constants we finally end up with the statement of this lemma. □

An analogous result to one of the main theorems of Chen and Kohn [21] can hence be formulated as

Theorem 10. *The function $\Phi(x)$ is an equilibrium price and the choice of the stopping time $\tau = t$ is optimal.*

Proof. Applying the Itô formula to $f(D_t, t) = \Phi(D_t)\, e^{-rt}$ results into

$$\begin{aligned} d\left(e^{-rt}\Phi(D_t)\right) &= -re^{-rt}\Phi(D_t)\,dt + e^{-rt}\Phi'(D_t)\,dD_t + \frac{1}{2}e^{-rt}\Phi''(D_t)\,(dD_t)^2 \\ &= e^{-rt}\big(-r\Phi(D_t)\,dt \\ &\quad +\Phi'(D_t)\left(\lambda_i(\mu - D_t)\,dt + \sigma\sqrt{D_t}dW_t^i\right) + \frac{1}{2}\Phi''(D_t)\,\sigma^2 D_t dt\big) \\ &= e^{-rt}\Big(\left(\lambda_i(\mu - D_t)\,\Phi'(D_t) + \frac{\sigma^2}{2}D_t\Phi''(D_t) - r\Phi(D_t)\right)dt \\ &\quad +\sigma^2\sqrt{D_t}\Phi'(D_t)\,dW_t^i\Big) \\ &\leq e^{-rt}(-D_t)\,dt + \sigma^2 e^{-rt}\sqrt{D_t}\Phi'(D_t)\,dW_t^i \end{aligned}$$

for $i = 1, 2$. The inequality holds due to Lemma 7. Let $\tau \geq t$ be a stopping time. Using

$$\mathbb{E}^{P^i}\left(\int_t^\tau e^{-ru} D_u \Phi'(u) dW_u^i \middle| D_t = x\right) = 0, \quad (3.16)$$

we obtain the inequality

$$\mathbb{E}^{P^i}\left(e^{-r\tau}\Phi\left(D_\tau\right)\middle|D_t=x\right)\le e^{-rt}\Phi\left(x\right)+\mathbb{E}^{P^i}\left(\int_t^\tau e^{-ru}\left(-D_u\right)du\middle|D_t=x\right) \tag{3.17}$$

and by rearranging

$$\mathbb{E}^{P^i}\left(\int_t^\tau e^{-ru}\left(D_u\right)du+e^{-r\tau}\Phi\left(D_\tau\right)\middle|D_t=x\right)\le e^{-rt}\Phi\left(x\right). \tag{3.18}$$

This inequality holds taking the maximum over $i=1,2$ and the supremum over the stopping times and we receive

$$\Phi\left(x\right)\ge\max_{i=1,2}\sup_{\tau\ge t}\mathbb{E}^{P^i}\left(\int_t^\tau e^{-r(u-t)}\left(D_u\right)du+e^{-r(\tau-t)}\Phi\left(D_\tau\right)\middle|D_t=x\right). \tag{3.19}$$

This holds with equality choosing $\tau=t$ because of

$$\Phi\left(x\right)\ge\max_{i=1,2}\sup_{\tau\ge t}\mathbb{E}^{P^i}\left(\int_t^\tau e^{-r(u-t)}\left(D_u\right)du+e^{-r(\tau-t)}\Phi\left(D_\tau\right)\middle|D_t=x\right)\ge\Phi\left(x\right). \tag{3.20}$$

Since $\Phi(x)=O(x)$, for the stopping time $\tau=N$ with $N\to\infty$ results

$$\Phi\left(x\right)\ge\max_{i=1,2}\mathbb{E}^{P^i}\left(\int_t^\infty e^{-r(u-t)}\left(D_u\right)du\middle|D_t=x\right)=I\left(x\right). \tag{3.21}$$

Hence, $\Phi\left(x\right)\ge I\left(x\right)$ and

$$\Phi\left(x\right)=\max_{i=1,2}\sup_{\tau\ge t}\mathbb{E}^{P^i}\left(\int_t^\tau e^{-r(u-t)}\left(D_u\right)du+e^{-r(\tau-t)}\Phi\left(D_\tau\right)\middle|D_t=x\right) \tag{3.22}$$

hold. Therefore, $\Phi\left(x\right)$ is an equilibrium price and the choice $\tau=t$ is optimal. □

With a link to the theory of viscosity solutions we now show that $\Phi\left(x\right)$ is in fact the minimal equilibrium price. The ideas stay completely the same as in [21, 22], but the proofs undergo some minor changes. Let $P_*\left(x\right)$ denote an iteratively constructed minimal equilibrium price analogous to [21].

Lemma 8. *$\Phi\left(x\right)\le P_*\left(x\right)$ and $\lim_{x\to\infty}\Phi\left(x\right)-P_*\left(x\right)=0$ hold.*

Proof. According to Theorem 10 the function $\Phi\left(x\right)$ is an equilibrium price the function $P_*\left(x\right)$ is a minimal equilibrium price. Obviously follows $\Phi\left(x\right)\le P_*\left(x\right)$. By construction is $P_*\left(x\right)\ge I\left(x\right)$. For $x\ge\mu$, we can write

$$\Phi(x)=\mathrm{U}\left(\frac{r}{\lambda_1},\frac{2\lambda_1\mu}{\sigma^2},\frac{2\lambda_1}{\sigma^2}x\right)C_2+I(x). \tag{3.23}$$

Apparently we obtain $\lim_{x\to\infty}\Phi\left(x\right)-I\left(x\right)=0$.

$$0\le\lim_{x\to\infty}\Phi\left(x\right)-P_*\left(x\right)\le\lim_{x\to\pm\infty}\Phi\left(x\right)-I\left(x\right)=0. \tag{3.24}$$

□

As have shown Chen and Kohn [21], the minimal equilibrium price $P_*\left(x\right)$ is lower semicontinuous following the same argumentation. In almost the same way, it can be shown in a long technical proof with very small changes from [21] that

Lemma 9. *$P_*(x)$ is a viscosity supersolution of*

$$-\left(\max\left(\lambda_1\left(\mu-x\right),\lambda_2\left(\mu-x\right)\right)u'\left(x\right)+\frac{\sigma^2}{2}xu''\left(x\right)-ru\left(x\right)+x\right)=0. \tag{3.25}$$

Theorem 11. *$\Phi(x) = P_*(x)$ is the minimal equilibrium price.*

Proof. We proceed almost analogously to the proof in the erratum by Kohn [22]. Due to Theorem 10, the function $\Phi(x)$ is an equilibrium price. So, $P_*(x) \leq \Phi(x)$ clearly holds and we just have to show $P_*(x) \geq \Phi(x)$. Consider $\inf_{x\in\mathbb{R}}\{P_*(x) - \Phi(x)\}$. The minimum is zero for $x \to \infty$ according to Lemma 8. Recall that $\Phi(x)$ is continuous and $P_*(x)$ lower semicontinuous. If x stays bounded, there exists a point $\hat{x}$ where $P_*(x) - \Phi(x)$ takes its minimum. On the one hand $\Phi(x)$ fulfils the differential equation

$$\max\left(\lambda_1\left(\mu-\hat{x}\right),\lambda_2\left(\mu-\hat{x}\right)\right)\Phi'\left(\hat{x}\right)+\frac{\sigma^2}{2}\hat{x}\Phi''\left(\hat{x}\right)-r\Phi\left(\hat{x}\right)+\hat{x}=0 \tag{3.26}$$

and on the other hand $P_*(x)$ is according to Lemma 9 a viscosity supersolution which leads to

$$-\left(\max\left(\lambda_1\left(\mu-\hat{x}\right),\lambda_2\left(\mu-\hat{x}\right)\right)\Phi'\left(\hat{x}\right)+\frac{\sigma^2}{2}\hat{x}\Phi''\left(\hat{x}\right)-rP_*\left(\hat{x}\right)+\hat{x}\right)\geq 0. \tag{3.27}$$

Combining (3.26) and (3.27) we get

$$rP_*\left(\hat{x}\right)-r\Phi\left(\hat{x}\right)\geq 0 \tag{3.28}$$

and since $r > 0$ the desired result. □

The last theorem allows us to calculate the minimal equilibrium price and thus also the size of the asset bubble. The price bubble, defined as difference between minimal equilibrium price and intrinsic value, can be written as

$$B(x)=\begin{cases}\mathrm{M}\left(\frac{r}{\lambda_1},\frac{2\lambda_1\mu}{\sigma^2},\frac{2\lambda_1}{\sigma^2}x\right)C_1 & \text{for } x<\mu,\\ \mathrm{U}\left(\frac{r}{\lambda_2},\frac{2\lambda_2\mu}{\sigma^2},\frac{2\lambda_2}{\sigma^2}x\right)C_2 & \text{for } x\geq\mu\end{cases} \tag{3.29}$$

with the constants from Lemma 7. Concerning the properties, one can make very similar observations to [21]. For $\lambda_1 - \lambda_2 \to 0$ the constants C_1 and C_2 become zero and thus, the bubble will disappear.

The key insight from this chapter is that modifications of the model assumptions will have a big impact on the equilibrium price. The intrinsic value is more dependent on the structure of the equation.

4. A Regime Switching Equilibrium Model for Asset Bubbles

4.1. Model Setting

In this chapter we present a regime switching equilibrium model for asset bubbles based on [21]. We consider two investor groups $\iota = 1, 2$ with heterogeneous beliefs. Let $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^\iota\right)$ be a filtered probability space satisfying the usual hypothesis. We assume that there is one asset which pays a dividend D_t at time t described by the equation

$$dD_t = \lambda_\iota(\mu - D_t)dt + \sigma dW_t^\iota \tag{4.1}$$

with the parameters $\mu \geq 0$, $\sigma > 0$ and $\lambda_2 > \lambda_1 > 0$. Let W^1 be a $\left(\mathcal{F}^W, \mathbb{P}^1\right)$ Brownian motion. Then D is an Orstein-Uhlenbeck process for the first investor group. The interest rate shall be piecewise constant and switch between different states. For example, it could have just two states such as "good" or "bad" economic regime. Therefore, we proceed as in [41]. First, let $X = (X_t)_{t\geq 0}$ be a continuous homogeneous $\left(\mathcal{F}^X, \mathbb{P}^1\right)$ Markov chain in canonical representation on the state space of unit vectors $\{\mathtt{e}_1, \ldots, \mathtt{e}_N\}$ with rate matrix $\mathtt{A}^1$. Note that $\mathtt{A}^{1\intercal}$ is the usual generator matrix and hence a Metzler matrix. Let $\mathtt{r} = (r_1, \ldots, r_n)^\intercal \in \mathbb{R}^N$ and $0 < r_1 < \ldots < r_N < \infty$. We assume that the group 1 uses the interest rate $\rho_t = \langle \mathtt{r}, X_t \rangle$ at time $t \geq 0$. Consider the filtration $\mathcal{F}_t = \mathcal{F}^W \vee \mathcal{F}^X$ where $\mathcal{F}^W$ is the filtration generated by the Brownian motion and $\mathcal{F}^X$ by the Markov chain. The processes X and W^1 are supposed to be independent. Let $\mathtt{A}^2$ also be a rate matrix of the same dimension as $\mathtt{A}^1$. First, we want to find a measure under which X is an $\left(\mathcal{F}^X, \mathbb{P}^2\right)$ Markov chain and W^2 an $\left(\mathcal{F}^W, \mathbb{P}^2\right)$ Brownian motion. Therefore, we use a similar idea as in [41]. Define the process Λ^W by

$$d\Lambda_t^W = \exp\left(-\int_0^t \frac{\lambda_1 - \lambda_2}{\sigma}(\mu - D_s)\, dW_s^1 - \frac{1}{2}\int_0^t \left(\frac{\lambda_1 - \lambda_2}{\sigma}(\mu - D_s)\right)^2 ds\right). \tag{4.2}$$

We show that this is a martingale by verifying the Novikov condition (see Theorem 6). Thereby, we remember that the Ornstein-Uhlenbeck process is normally distributed with

$$D_t \sim N\left(\mu + e^{-\lambda_1 t}(D_0 - \mu), \frac{\sigma^2}{2\lambda_1}\left(1 - e^{-2\lambda_1 t}\right)\right) = N(m_t, s_t^2). \tag{4.3}$$

Let $c = \frac{1}{2} e^{\left(\frac{\lambda_2 - \lambda_1}{\sigma a}\right)^2}$ and choose $\varepsilon > 0$ and δ such that $0 < \delta \leq 1 - \varepsilon c s_t^2$ for all $t > 0$. Further, we define a sequence $T_n = n\varepsilon$. Applying Jensen's inequality and then Fubini's theorem results

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^1}\left(\exp\left(\frac{1}{2}\int_{T_n}^{T_n+\varepsilon}\left(\frac{\lambda_2 - \lambda_1}{\sigma a}\right)^2 D_t^2\right) dt\right) \\
&\quad \leq \mathbb{E}^{\mathbb{P}^1}\left(\frac{1}{\varepsilon}\int_{T_n}^{T_n+\varepsilon} \exp\left(\frac{\varepsilon}{2}\left(\frac{\lambda_2 - \lambda_1}{\sigma a}\right)^2 D_t^2\right) dt\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\varepsilon}\int_{T_n}^{T_n+\varepsilon} \mathbb{E}^{\mathbb{P}^1}\left(\exp\left(\frac{\varepsilon c}{2}D_t^2\right)\right)dt \\
&= \frac{1}{\varepsilon}\int_{T_n}^{T_n+\varepsilon} \exp\left(\frac{\varepsilon c m_t^2}{2-2c\varepsilon s_t^2}\right)\frac{1}{\sqrt{1-\varepsilon c s_t^2}}dt \\
&\leq \frac{1}{\varepsilon}\int_{T_n}^{T_n+\varepsilon} \exp\left(\frac{\varepsilon c m_t^2}{2\delta}\right)\frac{1}{\sqrt{\delta}}dt \leq \frac{1}{\varepsilon}\int_{T_n}^{T_n+\varepsilon}\frac{k}{\sqrt{\delta}}dt = \frac{k}{\sqrt{\delta}} < \infty.
\end{aligned}
$$

Hence, the Novikov condition holds (with Corollary 5.14 from [68, p. 199]) and Λ^W is a martingale. Now we consider the density process for the Markov chain. First, we have to introduce the notation (see [33], [41]). We write $\mathtt{A}^1 = (A^1_{ij})_{i,j=1,\dots,N}$ and $\mathtt{a}^1 = (A^1_{11}, A^1_{22}, \dots, A^1_{NN})^\intercal$. For any vector v we define with $\operatorname{diag}(v)$ the matrix which has the vector v in its diagonal. Let $\mathtt{I}$ denote the N-dimensional unit matrix and $\mathtt{1}$ an N-dimensional vector with 1 in each entry. Define the matrices $\mathtt{A}^1_0 = \mathtt{A}^1 - \operatorname{diag}(\mathtt{a}^1)$ and $\mathtt{B} = (B_{ij})_{i,j=1,\dots,N}$ where

$$
B_{ij} = \begin{cases} \frac{A^2_{ij}}{A^1_{ij}} & \text{if } A^1_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}
$$

The process $\mathbf{N}$ where

$$
\mathbf{N}_t = \int_0^t \left(\mathtt{I} - \operatorname{diag}\left(X_{s-}\right)\right) dX_s \tag{4.5}
$$

counts in its i-th entry the number of times the process X jumps to $\mathtt{e}_i$ in the interval $[0,t]$ from any other state. This notation allows us to write

$$
X_t = X_0 + \int_0^t \left(\mathtt{I} - X_{s-}\mathtt{1}^\intercal\right) d\mathbf{N}_s. \tag{4.6}
$$

Now we can define a process Λ^X by

$$
\Lambda^X_t = 1 + \int_0^t \Lambda^X_{s-}\left(\mathtt{B}X_{s-} - \mathtt{1}\right)^\intercal \left(d\mathbf{N}_s - \mathtt{A}^1_0 X_{s-} ds\right) \tag{4.7}
$$

and $\Lambda^X_0 = 1$. With the martingale representation of X (see [35]), we get

$$
\begin{aligned}
d\mathbf{N}_t &= \left(\mathtt{I} - \operatorname{diag}\left(X_{t-}\right)\right) dX_t \\
&= \left(\mathtt{I} - \operatorname{diag}\left(X_{t-}\right)\right)\mathtt{A}^1 X_t dt + \left(\mathtt{I} - \operatorname{diag}\left(X_{t-}\right)\right) dM^1_t \\
&= \mathtt{A}^1_0 X_t dt + \left(\mathtt{I} - \operatorname{diag}\left(X_{t-}\right)\right) dM^1_t.
\end{aligned}
$$

Therefore, $\mathbf{N}_t - \int_0^t \mathtt{A}^1_0 X_s ds$ is an $\left(\mathcal{F}^X_t, \mathbb{P}^1\right)$ martingale. Thus, Λ^X is a martingale. Finally, we define a process Λ where $\Lambda_t = \Lambda^X_t \Lambda^W_t$. Due to

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}^1}\left(\Lambda_t | \mathcal{F}_s\right) &= \mathbb{E}^{\mathbb{P}^1}\left(\mathbb{E}^{\mathbb{P}^1}\left(\Lambda^W_t | \mathcal{F}^W_s\right)\Lambda^X_t \middle| \mathcal{F}_s\right) \\
&= \Lambda^W_s \mathbb{E}^{\mathbb{P}^1}\left(\mathbb{E}^{\mathbb{P}^1}\left(\Lambda^X_t | \mathcal{F}^X_s\right)\middle| \mathcal{F}_s\right) = \Lambda_s,
\end{aligned}
$$

the process Λ is an $\left(\mathcal{F}_t, \mathbb{P}^1\right)$ martingale. Let us now consider new measure $\mathbb{P}^2$ defined by

$$
\left.\frac{d\mathbb{P}^2}{d\mathbb{P}^1}\right|_{\mathcal{F}_t} = \Lambda_t. \tag{4.8}
$$

Applying the Itô formula to $f\left(\Lambda_t^X\right) = \log\left(\Lambda_t^X\right)$, we get

$$\begin{aligned}\log\left(\Lambda_t^X\right) &= \int_0^t \frac{\Lambda_{s-}^X}{\Lambda_{s-}^X} \left(\mathtt{B}X_{s-} - \mathtt{1}\right)^\intercal \Lambda_t^X + \sum_{0<s\le t}\left(\log\left(\Lambda_s^X\right) - \log\left(\Lambda_{s-}^X\right) - \frac{\Delta\Lambda_{s-}^X}{\Lambda_{s-}^X}\right)\\ &= \int_0^t \left(\mathtt{B}X_{s-} - \mathtt{1}\right)^\intercal \left(d\mathbf{N}_s - \mathtt{A}_0^1 X_s ds\right) + \sum_{0<s\le t}\left(1 + \left(\mathtt{B}X_{s-} - \mathtt{1}\right)^\intercal \Delta\mathbf{N}_s\right).\end{aligned}$$

Thus, we can write the density for the measure change as

$$\begin{aligned}\Lambda_t = \exp\Bigg(&-\int_0^t \frac{\lambda_1-\lambda_2}{\sigma}\left(\mu - D_s\right) dW_s^1 - \frac{1}{2}\int_0^t \left(\frac{\lambda_1-\lambda_2}{\sigma}\left(\mu - D_s\right)\right)^2 ds\\ &- \int_0^t \mathtt{1}^\intercal\left(\mathtt{A}_0^2 - \mathtt{A}_0^1\right) X_{s-} ds\Bigg) \prod_{0<s\le t}\left(1 + \left(\mathtt{B}X_{s-} - \mathtt{1}\right)^\intercal \Delta\mathbf{N}_s\right). \quad (4.9)\end{aligned}$$

By Girsanov's theorem (see Theorem 7) the process W^2 defined by

$$dW_t^2 = dW_t^1 - \frac{\lambda_2-\lambda_1}{\sigma}\left(\mu - D_t\right) dt, \quad (4.10)$$

is an $\left(\mathcal{F}_t^W, \mathbb{P}^2\right)$ Brownian motion. Hence, D is an Ornstein-Uhlenbeck process for the second investor group. A similar result holds also for the Markov chain after the measure change.

Lemma 10. *The process X is an $\left(\mathcal{F}_t^X, \mathbb{P}^2\right)$ Markov chain with rate matrix $\mathtt{A}^2$.*

Proof. (see Lemma 2.3 in [33]) We show that

$$\bar{M}_t = \mathbf{N}_t - \int_0^t \mathtt{A}_0^2 X_s ds \quad (4.11)$$

is an $\mathcal{F}_t^X$ martingale with respect to $\mathbb{P}^2$. Therefore, we have to show that $\Lambda_t \bar{M}_t$ is an $\mathcal{F}_t^X$ martingale with respect to $\mathbb{P}^1$. Using the definition of the processes Λ and $\mathbf{N}$ we obtain

$$\begin{aligned}\Lambda_t^X \bar{M}_t &= \int_0^t \Lambda_{s-}^X d\bar{M}_s + \int_0^t \bar{M}_{s-} d\Lambda_s^X + \left[\Lambda^X, \bar{M}\right]_t\\ &= \int_0^t \Lambda_{s-}^X d\mathbf{N}_s - \int_0^t \Lambda_{s-}^X \mathtt{A}_0^2 X_s ds + \int_0^t \bar{M}_{s-} d\Lambda_s^X\\ &\quad + \sum_{0<s\le t} \Lambda_{s-}^X \left(\mathtt{B}_0 X_{s-} - \mathtt{1}\right)^\intercal \Delta\mathbf{N}_s \Delta\mathbf{N}_s\\ &= \int_0^t \Lambda_{s-}^X d\mathbf{N}_s - \int_0^t \Lambda_{s-}^X \mathtt{A}_0^2 X_s ds + \int_0^t \bar{M}_{s-} d\Lambda_s^X\\ &\quad + \int_0^t \Lambda_{s-}^X \operatorname{diag}\left(d\mathbf{N}_s\right)\left(\mathtt{B}_0 X_{s-} - \mathtt{1}\right)\\ &= \int_0^t \Lambda_{s-}^X d\mathbf{N}_s - \int_0^t \Lambda_{s-}^X \mathtt{A}_0^2 X_s ds + \int_0^t \bar{M}_{s-} d\Lambda_s^X\\ &\quad + \int_0^t \Lambda_{s-}^X \operatorname{diag}\left(\mathtt{A}_0^1 X_s\right)\left(\mathtt{B}_0 X_{s-} - \mathtt{1}\right) ds\\ &\quad + \int_0^t \Lambda_{s-}^X \operatorname{diag}\left(d\mathbf{N}_s - \mathtt{A}_0^1 X_s ds\right)\left(\mathtt{B}_0 X_{s-} - \mathtt{1}\right).\end{aligned}$$

Since for $X_t = \mathtt{e}_i$ we have $A^1_{ij} X_t(i)\,(b_{ij}-1) = \left(A^2_{ii}X_s(i) - A^1_{ii}X_s(i)\right)$, it is obvious that $\mathtt{A}^1_0 X_t\,(\mathtt{B}-\mathtt{1}) = \left(\mathtt{A}^2_0 X_s - \mathtt{A}^1_0 X_s\right)$ holds. Hence, we obtain

$$\begin{aligned}\Lambda^X_t \bar{M}_t = &\int_0^t \Lambda^X_{s-}\left(d\mathbf{N}_s - \mathtt{A}^1_0 X_s ds\right) + \int_0^t \bar{M}_{s-} d\Lambda^X_s \\ &+ \int_0^t \Lambda^X_{s-}\operatorname{diag}\left(d\mathbf{N}_s - \mathtt{A}^1_0 X_s ds\right)\left(\mathtt{B}_0 X_{s-} - \mathtt{1}\right).\end{aligned}$$

Obviously, $\Lambda^X_t \bar{M}_t$ is an $\left(\mathcal{F}^X_t, \mathbb{P}^1\right)$ martingale. Finally, using the definition of $\mathtt{A}^2_0$ we receive

$$\begin{aligned}X_t &= X_0 + \int_0^t \left(\mathtt{I} - X_{s-}\mathtt{1}^\intercal\right) d\mathbf{N}_s \\ &= X_0 + \int_0^t \left(\mathtt{I} - X_{s-}\mathtt{1}^\intercal\right)\mathtt{A}^2_0 X_s ds + \int_0^t \left(\mathtt{I} - X_{s-}\mathtt{1}^\intercal\right)\left(d\mathbf{N}_s - \mathtt{A}^2_0 X_s ds\right) \\ &= X_0 + \int_0^t \mathtt{A}^2 X_s ds + M^2_t\end{aligned}$$

where M^2_t is an $\left(\mathcal{F}^X_t, \mathbb{P}^2\right)$ martingale. □

We have seen that the investor groups can even disagree on the transition probabilities of the Markov chain. Further, note that we are now discounting with a non-constant interest rate. Therefore, we need the following lemma.

Lemma 11. *Let $s \geq t$. The expected discounted value of one monetary unit at time t can be expressed as*

$$\mathbb{E}^{\mathbb{P}^\iota}\left(e^{-\int_t^s \langle \mathtt{r}, X_u\rangle du}\middle|\mathcal{F}^X_t\right) = \left\langle e^{(s-t)(\mathtt{A}^\iota - \operatorname{diag}(\mathtt{r}))} X_t, \mathtt{1}\right\rangle. \tag{4.12}$$

Proof. The idea is taken from the proof of Theorem 4 in [91]. First, we recall $\rho_t = \langle \mathtt{r}, X_t\rangle$. We fix $t > 0$ and define a process

$$\Gamma_s = \exp\left(-\int_t^s \rho_u du\right) X_s. \tag{4.13}$$

Let us recall the semi-martingale-representation of X shown in [35]. It allows us the decomposition

$$dX_t = \mathtt{A}^\iota X_t dt + dM^\iota_t \tag{4.14}$$

where M^ι is a martingale under the measure $\mathbb{P}^\iota$ with respect to the filtration generated by X. With this we obtain

$$d\Gamma_s = -\rho_s \Gamma_s ds + \exp\left(-\int_t^s \rho_u du\right)\left(\mathtt{A}^\iota X_s ds + dM^\iota_s\right) \tag{4.15}$$

and therefore

$$\Gamma_s = \Gamma_t - \int_t^s \rho_v \Gamma_v dv + \int_t^s \exp\left(-\int_t^v \rho_u du\right)\left(\mathtt{A}^\iota X_v dv + dM^\iota_v\right). \tag{4.16}$$

Taking the expectation leads to

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_s \middle| \mathcal{F}^X_t\right) = \\ &= \Gamma_t - \int_t^s \rho_v \mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_v \middle| \mathcal{F}^X_t\right) dv + \int_t^s \mathbb{E}^{\mathbb{P}^\iota}\left(\exp\left(\int_t^v \rho_u du\right)\middle|\mathcal{F}^X_t\right)\mathtt{A}^\iota X_v dv \\ &= \Gamma_t + \int_t^s \left(\mathtt{A}^\iota - \operatorname{diag}(\mathtt{r})\right)\mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_v \middle| \mathcal{F}^X_t\right) dv.\end{aligned}$$

Fixing $t \geq 0$, we can solve

$$\mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_s \middle| \mathcal{F}_t^X\right) = \Gamma_t + \int_t^s \left(\mathtt{A}^\iota - \operatorname{diag}(\mathtt{r})\right) \mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_v \middle| \mathcal{F}_t^X\right) dv \tag{4.17}$$

for $\mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_s \middle| \mathcal{F}_t^X\right)$ and hence obtain

$$\mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_s \middle| \mathcal{F}_t^X\right) = \exp\left((s-t)\left(\mathtt{A}^\iota - \operatorname{diag}(\mathtt{r})\right)\right) X_s. \tag{4.18}$$

Finally, using the fact that

$$\exp\left(-\int_t^s \rho_u du\right) = \langle \Gamma_s, \mathtt{1} \rangle \tag{4.19}$$

we receive

$$\mathbb{E}^{\mathbb{P}^\iota}\left(\exp\left(-\int_t^s \langle \mathtt{r}, X_u \rangle \, du\right) \middle| \mathcal{F}_t^X\right) = \left\langle \mathbb{E}^{\mathbb{P}^\iota}\left(\Gamma_s \middle| \mathcal{F}_t^X\right), \mathtt{1} \right\rangle. \tag{4.20}$$

□

4.2. Intrinsic Value and Equilibrium Price

The *intrinsic value* at time $t \geq 0$ is defined as the maximal price, an investor is willing to pay for all expected discounted future dividends, i.e.

$$I(\mathtt{x}, y) = \max_{\iota=1,2} \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\infty e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} D_s ds \middle| D_t = y, X_t = \mathtt{x}\right) \tag{4.21}$$

given the current state of the economy $\mathtt{x} \in \{\mathtt{e}_1, \ldots, \mathtt{e}_N\}$.

Lemma 12. *The intrinsic value can be written as*

$$I(\mathtt{x}, y) = \max_{\iota=1,2} \mathtt{x}^\intercal I_\iota(y) \tag{4.22}$$

where $\mathtt{K}^\iota = \mathtt{A}^\iota - \operatorname{diag}(\mathtt{r})$ *and*

$$I_\iota(y) = \left(\left(\lambda_\iota \mathtt{I} - \mathtt{K}^\iota\right)^{-1} y - \left(\mathtt{K}^\iota\right)^{-1}\left(\lambda_\iota \mathtt{I} - \mathtt{K}^\iota\right)^{-1} \mu \lambda_\iota\right)^\intercal \mathtt{1}. \tag{4.23}$$

Proof. Since the distribution of an Ornstein-Uhlenbeck process is well-known, we receive

$$\begin{aligned}
I(\mathtt{x}, y) &= \max_{\iota=1,2} \int_t^\infty \mathbb{E}^{\mathbb{P}^\iota}\left(e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} D_s \middle| D_t = y, X_t = \mathtt{x}\right) ds \\
&= \max_{\iota=1,2} \int_t^\infty \mathbb{E}^{\mathbb{P}^\iota}\left(\mathbb{E}^{\mathbb{P}^\iota}\left(e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} \middle| X_t = \mathtt{x}\right) D_s \middle| D_t = y, X_t = \mathtt{x}\right) ds \\
&= \max_{\iota=1,2} \int_t^\infty \mathbb{E}^{\mathbb{P}^\iota}\left(e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} \middle| X_t = \mathtt{x}\right) \mathbb{E}^{\mathbb{P}^\iota}\left(D_s \middle| D_t = y\right) ds \\
&= \max_{\iota=1,2} \int_t^\infty \mathbb{E}^{\mathbb{P}^\iota}\left(e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} \middle| X_t = \mathtt{x}\right) \left(\mu + e^{-\lambda_\iota (s-t)}(y - \mu)\right) ds
\end{aligned}$$

using the independence of X and D. With Lemma 11 we obtain

$$I(\mathtt{x}, y) = \max_{\iota=1,2} \int_t^\infty \left\langle e^{(s-t)(\mathtt{A}^\iota - \operatorname{diag}(\mathtt{r}))} \mathtt{x} \left(\mu + e^{-\lambda_\iota (s-t)}(y - \mu)\right), \mathtt{1} \right\rangle ds. \tag{4.24}$$

This integral can be computed explicitly. Let $\mathtt{L}$ be a non-singular $N \times N$ matrix and $c \geq t$. First, we compute

$$\int_t^c e^{(s-t)\mathtt{L}} ds = \int_t^c \sum_{k=0}^{\infty} \frac{1}{k!} (s-t)^k \mathtt{L}^k ds = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} (s-t)^{k+1} \mathtt{L}^k \Bigg|_{s=t}^{c}$$
$$= \mathtt{L}^{-1} \left(-\mathtt{I} + \sum_{k=0}^{\infty} \frac{1}{k!} (c-t)^k \mathtt{L}^k \right) = -\mathtt{L}^{-1} + \mathtt{L}^{-1} e^{(c-t)\mathtt{L}}.$$

Now we show that for $\mathtt{L} = \mathtt{K}^\iota$ and for $\mathtt{L} = \mathtt{K}^\iota - \lambda_\iota \mathtt{I}$, the integral converges for $c \to \infty$. There are several methods to compute a matrix exponential [79]. We take a look at the complex Jordan decomposition of the matrix $\mathtt{L} = \mathtt{T}\mathtt{J}\mathtt{T}^{-1}$. Note, that the case where this matrix is diagonalisable, can be interpreted a Jordan decomposition with block size one. We can always write $e^{c\mathtt{L}} = \mathtt{T} e^{c\mathtt{J}} \mathtt{T}^{-1}$. Since $\mathtt{A}^\iota$ is a generator matrix,

$$\sum_{\substack{i=1 \\ i \neq j}}^{N} \left| K_{ij}^\iota \right| = \sum_{\substack{i=1 \\ i \neq j}}^{N} A_{ij}^\iota = -A_{ii}^\iota \tag{4.25}$$

holds. Due to the Geršgorin theorem, this is the radius of a Geršgorin cycle around $A_{ii}^\iota - r_i$ and hence the matrix $\mathtt{K}^\iota$ has only eigenvalues with negative real part. Analogously, it can be shown that the eigenvalues of $\mathtt{K}^\iota - \lambda_\iota \mathtt{I}$ have all negative real part. So the inverse of $\mathtt{K}^\iota$ and $\mathtt{K}^\iota - \lambda_\iota \mathtt{I}$ exist. The matrix exponential of a Jordan normal form can be computed explicitly [79]. Therefore, we observe that

$$\lim_{c \to \infty} e^{c\mathtt{L}} = \mathtt{T} \lim_{c \to \infty} e^{c\mathtt{J}} \mathtt{T}^{-1} = \mathtt{T}\mathtt{0}_{N,N}\mathtt{T}^{-1} = \mathtt{0}_{N,N}, \tag{4.26}$$

where $\mathtt{0}_{N,N}$ is an $N \times N$ zero matrix. Putting everything together, we receive

$$I(\mathtt{x}, y) = \max_{\iota=1,2} \left(\left(\mu \left(-\mathtt{K}^\iota \right)^{-1} + (y - \mu) \left(\lambda_\iota \mathtt{I} - \mathtt{K}^\iota \right)^{-1} \right) \mathtt{x} \right)^{\mathsf{T}} \mathtt{1}. \tag{4.27}$$

Using

$$\left(-\mathtt{K}^\iota \right)^{-1} = \left(\lambda_\iota \mathtt{I} - \mathtt{K}^\iota \right)^{-1} \left(\lambda_\iota \mathtt{I} - \mathtt{K}^\iota \right) \left(-\mathtt{K}^\iota \right)^{-1} \mu$$
$$= \left(\lambda_\iota \mathtt{I} - \mathtt{K}^\iota \right)^{-1} \left(-\mathtt{K}^\iota \right)^{-1} \lambda_\iota \mu + \left(\lambda_\iota \mathtt{I} - \mathtt{K}^\iota \right)^{-1} \mu$$

we get the desired representation. □

We restrict the model to liquid markets without short selling and transaction costs. An *equilibrium price* is defined as a continuous function satisfying

$$P(\mathtt{x}, y) = \max_{\iota=1,2} \sup_{\tau \geq t} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} D_s ds \right.$$
$$\left. + e^{-\int_t^\tau \langle \mathtt{r}, X_u \rangle du} P(X_\tau, D_\tau) \Big| X_t = \mathtt{x}, D_t = y \right) \tag{4.28}$$

and

$$P(\mathtt{x}, y) \geq I(\mathtt{x}, y) \tag{4.29}$$

where the supremum is taken over all stopping times (see [21]). Additionally, we assume $|P(D_\infty, y)| < \infty$. We assume that market participants trade only at equilibrium and maximise their expected wealth. The next result gives us a method to construct the minimal equilibrium price.

Theorem 12. *Define* $P_0(\mathtt{x}, y) = I(\mathtt{x}, y)$ *and*

$$P_k(\mathtt{x}, y) = \max_{\iota=1,2} \sup_{\tau \geq t} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} D_s ds + e^{-\int_t^\tau \langle \mathtt{r}, X_u \rangle du} P_{k-1}(X_\tau, D_\tau) \middle| X_t = \mathtt{x}, D_t = y \right) \tag{4.30}$$

for $k \geq 1$. *Then* $P_k(\mathtt{x}, y)$ *is monotonously increasing in* k *and*

$$P_*(\mathtt{x}, y) = \lim_{k \to \infty} P_k(\mathtt{x}, y) \tag{4.31}$$

is an equilibrium price and minimal.

Proof. The proof in the regime switching case is almost completely analogous to the original model (see [21]). Choosing the stopping time $\tau = t$ gives us

$$P_k(\mathtt{x}, y) \geq \max_{\iota=1,2} \mathbb{E}^{\mathbb{P}^\iota} \left(P_{k-1}(X_t, D_t) | X_t = \mathtt{x}, D_t = y \right) \geq P_{k-1}(\mathtt{x}, y). \tag{4.32}$$

Therefore the series $(P_k(\mathtt{x}, y))_{k \geq 0}$ is monotonously increasing. By the monotone convergence theorem, $P_*(\mathtt{x})$ is an equilibrium price.The minimality is shown by induction. Let $P(\mathtt{x}, y)$ be an equilibrium price. By definition, $P(\mathtt{x}, y) \geq I(\mathtt{x}, y)$ holds. Supposing $P(\mathtt{x}, y) \geq P_{k-1}(\mathtt{x}, y)$, we obtain

$$\begin{aligned} P(\mathtt{x}, y) &= \\ &\max_{\iota=1,2} \sup_{\tau \geq t} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-\int_t^s \rho_u du} D_s ds + e^{-\int_t^\tau \rho_u du} P(X_\tau, D_\tau) \middle| X_t = \mathtt{x}, D_t = y \right) \\ &\geq \max_{\iota=1,2} \sup_{\tau \geq t} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-\int_t^s \rho_u du} D_s ds + e^{-\int_t^\tau \rho_u du} P_{k-1}(X_\tau, D_\tau) \middle| X_t = \mathtt{x}, D_t = y \right) \\ &= P_k(\mathtt{x}, y). \end{aligned}$$

writing $\rho_u = \langle \mathtt{r}, X_u \rangle$. Taking the limit follows $P(\mathtt{x}, y) \geq \lim_{k\to\infty} P_k(\mathtt{x}, y) = P_*(\mathtt{x}, y)$. Thus, $P_*(\mathtt{x}, y)$ is a minimal equilibrium price. □

We have seen that the minimal equilibrium price depends on the initial state of the Markov chain and via the construction also on the generator matrix. The following result describes a fundamental property of equilibrium prices.

Theorem 13. *Let* P *be an equilibrium price. For every* $j \in \{1, \ldots, N\}$ *we define* $p_j(y) = P(r_j, y)$. *Then* $p_j(y)$ *satisfies the system of differential inequalities*

$$\lambda_\iota (\mu - y) p_j'(y) + \frac{\sigma^2}{2} p_j''(y) - r_j p_j(y) + \sum_{k=1}^N (p_k(y) - p_j(y)) A_{kj}^\iota + y \leq 0 \tag{4.33}$$

and $p_j(y) \geq I(\mathtt{e}_j, y)$ *for all* $j = 1, \ldots, N$ *and for* $\iota = 1, 2$.

Proof. First we apply the Itô formula with jumps onto

$$f(X_s, D_s, s) := e^{-\int_t^s \langle \mathtt{r}, X_u \rangle du} P(X_s, D_s). \tag{4.34}$$

Since X is a Markov chain, we receive

$$\begin{aligned} f(X_\tau, D_\tau, \tau) &= f(X_t, D_t, t) + \int_t^\tau f_y(X_{s-}, D_s, s) dD_s \\ &+ \int_t^\tau f_t(X_{s-}, D_s, s) ds + \frac{1}{2} \int_t^\tau f_{yy}(X_{s-}, D_s, s) d[D, D]_s^c \\ &+ \sum_{t < s \leq \tau} (f(X_s, D_s, s) - f(X_{s-}, D_s, s)) \end{aligned}$$

$$
\begin{aligned}
&= P\left(X_t, D_t\right) + \int_t^\tau e^{-\int_t^{s-}\langle \mathtt{r}, X_u\rangle du} P_y\left(X_{s-}, D_s\right) dW_s^\iota \\
&+ \int_t^\tau e^{-\int_t^{s-}\langle \mathtt{r}, X_u\rangle du} \Big(\lambda_\iota\left(\mu - D_s\right) P_y\left(X_{s-}, D_s\right) \\
&+ \frac{\sigma^2}{2} P_{yy}\left(X_{s-}, D_s\right) - \left\langle \mathtt{r}, X_{s-}\right\rangle P\left(X_{s-}, D_s\right)\Big) ds \\
&+ \sum_{t<s\leq\tau} \left(e^{-\int_t^{s}\langle \mathtt{r}, X_u\rangle du} P\left(X_s, D_s\right) - e^{-\int_t^{s}\langle \mathtt{r}, X_u\rangle du} P\left(X_{s-}, D_s\right)\right).
\end{aligned}
$$

Putting this into the definition of P we receive

$$
\begin{aligned}
P\left(\mathtt{x}, y\right) = \max_{\iota=1,2} \sup_{\tau\geq t} \mathbb{E}^{\mathbb{P}^\iota} \Bigg(& P\left(X_t, D_t\right) + \int_t^\tau e^{-\int_t^{s-}\langle \mathtt{r}, X_u\rangle du} P_y\left(X_{s-}, D_s\right) dW_s^\iota \\
&+ \int_t^\tau e^{-\int_t^{s-}\langle \mathtt{r}, X_u\rangle du} \Big(D_t + \lambda_\iota\left(\mu - D_s\right) P_y\left(X_{s-}, D_s\right) \\
&+ \frac{\sigma^2}{2} P_{yy}\left(X_{s-}, D_s\right) - \left\langle \mathtt{r}, X_{s-}\right\rangle P\left(X_{s-}, D_s\right)\Big) ds \\
&+ \sum_{t<s\leq\tau} e^{-\int_t^{s}\langle \mathtt{r}, X_u\rangle du} \left(P\left(X_s, D_s\right) - P\left(X_{s-}, D_s\right)\right) \Bigg| D_t = y, X_t = \mathtt{x} \Bigg).
\end{aligned}
$$

According to Lemma 11.2.3 from [14], the compensator of the jump part

$$
\sum_{t<s\leq\tau} e^{-\int_t^{s}\langle \mathtt{r}, X_u\rangle du} \left(P\left(X_s, D_s\right) - P\left(X_{s-}, D_s\right)\right)
$$

is

$$
\int_t^\tau \sum_{k=1}^N \left(P\left(\mathtt{e}_k, D_s\right) - P\left(X_{s-}, D_s\right)\right) \mathtt{e}_k^\intercal \mathtt{A}^\iota X_{s-} \mathbf{1}_{\left(X_{s-}\neq \mathtt{e}_k\right)} ds.
$$

Note that if X_{s-} is in state $\mathtt{e}_j$, the expression $\mathtt{e}_k^\intercal \mathtt{A}^\iota X_{s-}$ represents A_{kj}^ι. By taking the expectation, all terms integrated with respect to martingales are zero. Hence, we obtain

$$
\begin{aligned}
0 = \max_{\iota=1,2} \sup_{\tau\geq t} \mathbb{E}^{\mathbb{P}^\iota} \Bigg(& \int_t^\tau e^{-\int_t^{s-}\langle \mathtt{r}, X_u\rangle du} \Big(y + \lambda_\iota\left(\mu - D_s\right) P_y\left(X_{s-}, D_s\right) \\
&+ \frac{\sigma^2}{2} P_{yy}\left(X_{s-}, D_s\right) - \left\langle \mathtt{r}, X_{s-}\right\rangle P\left(X_{s-}, D_s\right)\Big) \\
&+ \sum_{k=1}^N \left(P\left(\mathtt{e}_k, D_s\right) - P\left(X_{s-}, D_s\right)\right) \mathtt{e}_k^\intercal \mathtt{A}^\iota X_{s-} ds \Bigg| D_t = y, X_t = \mathtt{x} \Bigg).
\end{aligned}
$$

The expression above can only be zero when the integrand is non-positive. Hence, we get

$$
\begin{aligned}
&\lambda_\iota\left(\mu - y\right) P_y\left(\mathtt{x}, y\right) + \frac{\sigma^2}{2} P_{yy}\left(\mathtt{x}, y\right) - \left\langle \mathtt{r}, \mathtt{x}\right\rangle P\left(\mathtt{x}, y\right) \\
&\qquad + \sum_{k=1}^N \left(P\left(\mathtt{e}_k, y\right) - P\left(\mathtt{x}, y\right)\right) \mathtt{e}_k^\intercal \mathtt{A}^\iota \mathtt{x} + y \leq 0. \quad (4.35)
\end{aligned}
$$

Rewriting this and using $P(\mathtt{x}, y) \geq I(\mathtt{x}, y)$, we have proven the theorem. □

Theorem 13 gives an idea how a minimal equilibrium price could look like. A function that satisfies the inequality with equality is still an equilibrium price. First we shall solve it and as a second step we identify a solution with a certain growth condition as minimal equilibrium price. Since (4.33) resembles a Weber equation, we use the power series method to find the solution. A similar approach has been studied for general hypergeometric matrix equations in [65, 67]. Appendix A provides the definitions and basic properties of some special matrix functions.

Lemma 13. *Consider the vector* $\Phi(y) := (\Phi_1(y), \dots, \Phi_N(y))^\mathsf{T}$ *and define* $\mathtt{K}^\iota = \mathtt{A}^\iota - \operatorname{diag}(\mathbf{r})$*. Let* $\mathtt{0}$ *be a vector of* N *zeros. Then the system of differential equations*

$$\frac{\sigma^2}{2}\Phi''(y) + \lambda_\iota\,(\mu - y)\,\Phi'(y) + (\mathtt{K}^\iota)^\mathsf{T}\,\Phi(y) + y\mathtt{1} = \mathtt{0} \tag{4.36}$$

has a solution

$$\begin{aligned}\Phi_\iota(y) = \mathrm{F}_{\frac{(\mathtt{K}^\iota)^\mathsf{T}}{\lambda_\iota}}\left(\frac{\sqrt{2\lambda_\iota}}{\sigma}(y-\mu)\right)\mathtt{k}_0^\iota + \mathrm{F}_{\frac{(\mathtt{K}^\iota)^\mathsf{T}}{\lambda_\iota}}\left(\frac{\sqrt{2\lambda_\iota}}{\sigma}(\mu-y)\right)\mathtt{k}_1^\iota \\ + \left(\left(\lambda_\iota\mathtt{I} - (\mathtt{K}^\iota)^\mathsf{T}\right)^{-1} y - \left((\mathtt{K}^\iota)^\mathsf{T}\right)^{-1}\left(\lambda_\iota\mathtt{I} - (\mathtt{K}^\iota)^\mathsf{T}\right)^{-1}\mu\lambda_\iota\right)\mathtt{1}.\end{aligned} \tag{4.37}$$

Proof. Analogously to the proof of Lemma 16, we can show using the Geršgorin theorem that the inverse of $(\mathtt{K}^\iota)^\mathsf{T}$ and $\lambda_\iota\mathtt{I} + (\mathtt{K}^\iota)^\mathsf{T}$ exist. Now we define

$$\phi(y) := \Phi(y) - \left(\lambda_\iota\mathtt{I} - (\mathtt{K}^\iota)^\mathsf{T}\right)^{-1} y\mathtt{1} + \left((\mathtt{K}^\iota)^\mathsf{T}\right)^{-1}\left(\lambda_\iota\mathtt{I} - (\mathtt{K}^\iota)^\mathsf{T}\right)^{-1}\mu\lambda_\iota\mathtt{1}. \tag{4.38}$$

Hence, the equation above turns into

$$\frac{\sigma^2}{2}\phi''(y) + \lambda_\iota\,(\mu - y)\,\phi'(y) + (\mathtt{K}^\iota)^\mathsf{T}\,\phi(y) = \mathtt{0}. \tag{4.39}$$

Let us now substitute $\tilde{y} = \frac{\sqrt{2\lambda_\iota}}{\sigma}(y-\mu)$ and $\psi(\tilde{y}) = \phi(y)$. Thus, we receive

$$\psi''(\tilde{y}) + \tilde{y}\psi'(\tilde{y}) + \frac{1}{\lambda_\iota}(\mathtt{K}^\iota)^\mathsf{T}\,\psi(\tilde{y}) = \mathtt{0}. \tag{4.40}$$

Now we construct a series representation of a solution. Hence, we assume the solution is of the form

$$\psi(\tilde{y}) = \sum_{k=0}^{\infty}\mathtt{c}_k\tilde{y}^k \tag{4.41}$$

with vectors $\mathtt{c}_k \in \mathbb{R}^N$ for $k \in \mathbb{N}$. Putting this in the equation, we obtain

$$\sum_{k=2}^{\infty} k(k-1)\mathtt{c}_k\tilde{y}^{k-2} - \sum_{k=1}^{\infty} k\mathtt{c}_k\tilde{y}^k + \sum_{k=0}^{\infty}\frac{1}{\lambda_\iota}(\mathtt{K}^\iota)^\mathsf{T}\,\mathtt{c}_k\tilde{y}^k = \mathtt{0} \tag{4.42}$$

and after rearranging

$$\sum_{k=0}^{\infty}\left((k+2)(k+1)\mathtt{c}_{k+2} - \left(k\mathtt{I} - \frac{1}{\lambda_\iota}(\mathtt{K}^\iota)^\mathsf{T}\right)\mathtt{c}_k\right)\tilde{y}^k = \mathtt{0}. \tag{4.43}$$

If the series is a solution, then all coefficients must be zero, so we have

$$(k+2)(k+1)\mathtt{c}_{k+2} - \left(k\mathtt{I} - \frac{1}{\lambda_\iota}(\mathtt{K}^\iota)^\mathsf{T}\right)\mathtt{c}_k = \mathtt{0} \tag{4.44}$$

for all $k \geq 0$. Therefore, we must solve the recurrence relation

$$\mathtt{c}_{k+2} = \frac{1}{(k+2)(k+1)} \left(k\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \mathtt{c}_k \tag{4.45}$$

separately for even and odd k. First we show

$$\mathtt{c}_{2k} = \frac{1}{(2k)!} \prod_{j=0}^{k-1} \left(2j\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \mathtt{c}_0 \tag{4.46}$$

by induction. Obviously, this is true for $k = 1$. Let us now assume this holds for $\mathtt{c}_k$. So we get

$$\begin{aligned} \mathtt{c}_{2(k+1)} &= \frac{1}{(2k+2)(2k+1)} \left(2k\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \frac{1}{(2k)!} \prod_{j=0}^{k-1} \left(2j\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \mathtt{c}_0 \\ &= \frac{1}{(2(k+1))!} \prod_{j=0}^{k} \left(2j\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \mathtt{c}_0. \end{aligned}$$

Analogously, we can show

$$\mathtt{c}_{2k+1} = \frac{1}{(2k+1)!} \prod_{j=0}^{k-1} \left((2j+1)\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \mathtt{c}_1. \tag{4.47}$$

Hence, we obtain

$$\begin{aligned} \psi(y) = \mathtt{c}_0 &+ \sum_{k=1}^{\infty} \frac{1}{(2k)!} \prod_{j=0}^{k-1} \left(2j\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \tilde{y}^{2k} \mathtt{c}_0 \\ &+ \tilde{y}\mathtt{c}_1 + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \prod_{j=0}^{k-1} \left((2j+1)\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) \tilde{y}^{2k+1} \mathtt{c}_1. \end{aligned}$$

Inductively, the relationship

$$\begin{aligned} (2k)! &= 2^{2k} \left(\frac{1}{2} \right)_k k!, \\ (2k+1)! &= 2^{2k} \left(\frac{3}{2} \right)_k k! \end{aligned}$$

can be easily shown. Moreover,

$$\begin{aligned} \prod_{j=0}^{k-1} \left(2j\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) &= 2^{2k} \left(-\frac{1}{2\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right)_k, \\ \prod_{j=0}^{k-1} \left((2j+1)\mathtt{I} - \frac{1}{\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right) &= 2^{2k} \left(\frac{1}{2}\mathtt{I} - \frac{1}{2\lambda_\iota} (\mathtt{K}^\iota)^\intercal \right)_k \end{aligned}$$

obviously holds, where we use the notation of a matrix Pochhammer symbol introduced in Appendix A. Putting everything together, we can write

$$\psi(y) = {}_1F_1\left(-\frac{(\mathtt{K}^\iota)^\intercal}{2\lambda_\iota}; \frac{1}{2}\mathtt{I}; \frac{\tilde{y}^2}{2} \right) \mathtt{c}_0 + |\tilde{y}| {}_1F_1\left(\frac{1}{2}\left(\mathtt{I} - \frac{(\mathtt{K}^\iota)^\intercal}{\lambda_\iota} \right); \frac{3}{2}\mathtt{I}; \frac{\tilde{y}^2}{2} \right) \mathtt{c}_1. \tag{4.48}$$

Analogously to the case without regime switching, we will not choose this solution. Putting $\psi(-\tilde{y})$ into (4.40), we can observe that it also solves the equation. With the proper choice of the constants, we rewrite the solution as linear combination of two second Kummer functions (see Appendix A for the definitions). After resubstituting we finally receive the desired representation. □

4.3. The Equilibrium Price as Solution of a Differential Equation

In the next step, our aim is to show that a continuous differentiable solution of

$$\max_{\iota\in\{1,2\}} \frac{\sigma^2}{2}\Phi''(y) + \lambda_\iota\,(\mu - y)\,\Phi'(y) + (\mathtt{K}^\iota)^\intercal\,\Phi(y) + y\mathtt{1} = \mathtt{0} \tag{4.49}$$

with linear growth at infinity is a minimal equilibrium price. For simplicity, we assume that both investor groups agree on the same rate matrix for the regime switching, although the comparison result from viscosity solution theory holds in general. The following theorem characterises such a solution of (4.49). Since $\mathtt{A}^1 = \mathtt{A}^2$, it follows $\mathtt{K}^1 = \mathtt{K}^2$. In order to shorten the notation, we write from now on

$$I_\iota(y) = \Big((\lambda_\iota\mathtt{I} - \mathtt{K})^{-1}\,y - \mathtt{K}^{-1}\,(\lambda_\iota\mathtt{I} - \mathtt{K})^{-1}\,\mu\lambda_\iota\Big)^\intercal\,\mathtt{1}$$
$$\tilde{I}_\iota(y) = \Big((\lambda_\iota\mathtt{I} - \mathtt{K}^\intercal)^{-1}\,y - (\mathtt{K}^\intercal)^{-1}\,(\lambda_\iota\mathtt{I} - \mathtt{K}^\intercal)^{-1}\,\mu\lambda_\iota\Big)\,\mathtt{1}.$$

Obviously, without regime switching $I_\iota(y)$ and $\tilde{I}_\iota(y)$ are identical.

Theorem 14. *If $\mathtt{A}^1 = \mathtt{A}^2$, then the differential equation*

$$\frac{\sigma^2}{2}\Phi''(y) + \max_{\iota\in\{1,2\}} \lambda_\iota\,(\mu - y)\,\Phi'(y) + \mathtt{K}^\intercal\Phi(y) + y\mathtt{1} = \mathtt{0} \tag{4.50}$$

has a unique continuous differentiable solution $\Phi(y)$ with at most linear growth at infinity. It can be determined explicitly as

$$\Phi(y) = \begin{cases} \mathrm{F}_{\frac{\mathtt{K}^\intercal}{\lambda_2}}\left(\frac{\sqrt{2\lambda_2}}{\sigma}(\mu - y\right)\mathtt{k}_2 + \tilde{I}_2(y) & \text{for } y \le \mu, \\ \mathrm{F}_{\frac{\mathtt{K}^\intercal}{\lambda_1}}\left(\frac{\sqrt{2\lambda_1}}{\sigma}(y - \mu\right)\mathtt{k}_1 + \tilde{I}_1(y) & \text{for } y \ge \mu, \end{cases} \tag{4.51}$$

with the constants

$$\mathtt{k}_1 = (\mathtt{F}_1)^{-1}\,(\mathtt{C}_1 + \mathtt{F}_2\mathtt{k}_2)\,,$$
$$\mathtt{k}_2 = \Big(\mathtt{F}_3\,(\mathtt{F}_1)^{-1}\,\mathtt{F}_2 - \mathtt{F}_4\Big)^{-1}\,\Big(\mathtt{C}_2 - \mathtt{F}_3\,(\mathtt{F}_1)^{-1}\,\mathtt{C}_1\Big)$$

where

$$\mathtt{F}_1 = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\left(\mathtt{I} - \frac{(\mathtt{K})^\intercal}{\lambda_1}\right)\right)^{-1},$$
$$\mathtt{F}_2 = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\left(\mathtt{I} - \frac{(\mathtt{K})^\intercal}{\lambda_2}\right)\right)^{-1},$$
$$\mathtt{F}_3 = \Gamma\left(-\frac{1}{2}\right)\frac{\sqrt{\lambda_1}}{\sigma}\Gamma\left(-\frac{(\mathtt{K})^\intercal}{2\lambda_1}\right)^{-1},$$
$$\mathtt{F}_4 = -\Gamma\left(-\frac{1}{2}\right)\frac{\sqrt{\lambda_2}}{\sigma}\Gamma\left(-\frac{(\mathtt{K})^\intercal}{2\lambda_2}\right)^{-1},$$
$$\mathtt{C}_1 = \tilde{I}_2(\mu) - \tilde{I}_1(\mu),$$
$$\mathtt{C}_2 = \Big((\lambda_2\mathtt{I} - \mathtt{K}^\intercal)^{-1} - (\lambda_1\mathtt{I} - \mathtt{K}^\intercal)^{-1}\Big)\,\mathtt{1}.$$

Proof. Combining the simplification $\mathtt{A}^1 = \mathtt{A}^2$ with the fact $\lambda_2 > \lambda_1$ and Theorem 13, we are able to write the solution of (4.50) as

$$\Phi(y) = \begin{cases} \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_2}}\left(\frac{\sqrt{2\lambda_2}}{\sigma}(y-\mu)\right)\mathtt{k}_1^2 + \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_2}}\left(\frac{\sqrt{2\lambda_2}}{\sigma}(\mu-y)\right)\mathtt{k}_2^2 + \tilde{I}_2(y) & \text{for } y \leq \mu, \\ \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_1}}\left(\frac{\sqrt{2\lambda_1}}{\sigma}(y-\mu)\right)\mathtt{k}_1^1 + \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_1}}\left(\frac{\sqrt{2\lambda_1}}{\sigma}(\mu-y)\right)\mathtt{k}_2^1 + \tilde{I}_1(y) & \text{for } y > \mu. \end{cases} \tag{4.52}$$

Due to Appendix A, we know that

$$\lim_{y\to\infty} \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_\iota}}\left(\frac{\sqrt{2\lambda_\iota}}{\sigma}(y-\mu)\right) = \mathtt{0} \tag{4.53}$$

and

$$\lim_{y\to-\infty} \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_\iota}}\left(\frac{\sqrt{2\lambda_\iota}}{\sigma}(\mu-y)\right) = \mathtt{0} \tag{4.54}$$

for $\iota \in \{1, 2\}$. On the other hand,

$$\lim_{y\to-\infty} \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_\iota}}\left(\frac{\sqrt{2\lambda_\iota}}{\sigma}(y-\mu)\right) \quad \text{and} \quad \lim_{y\to\infty} \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_\iota}}\left(\frac{\sqrt{2\lambda_\iota}}{\sigma}(\mu-y)\right) \tag{4.55}$$

do not converge. Therefore, we set $\mathtt{k}_1^2 = \mathtt{k}_2^1 = \mathtt{0}$ and rename $\mathtt{k}_1^1 = \mathtt{k}_1$ and $\mathtt{k}_2^2 = \mathtt{k}_2$. As we want the solution and its derivative to be continuous at $y = \mu$, we take the limit from both sides and solve

$$\begin{aligned} \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_1}}(0)\,\mathtt{k}_1 - \mathrm{F}_{\frac{\mathtt{KT}}{\lambda_2}}(0)\,\mathtt{k}_2 &= \tilde{I}_2(\mu) - \tilde{I}_1(\mu), \\ \mathrm{F}'_{\frac{\mathtt{KT}}{\lambda_1}}(0)\,\mathtt{k}_1 - \mathrm{F}'_{\frac{\mathtt{KT}}{\lambda_2}}(0)\,\mathtt{k}_2 &= \left((\lambda_2\mathtt{I} - \mathtt{K})^{-1} - (\lambda_1\mathtt{I} - \mathtt{K})^{-1}\right)\mathtt{1} \end{aligned}$$

to determine the coefficients $\mathtt{k}_1$ and $\mathtt{k}_2$ uniquely. Note that one can easily show that all matrices we need to invert are non-singular. □

Now we will show that $\Phi(y)$ from Theorem 14 is a minimal equilibrium price. Therefore, we proceed state-wise analogously to [21, 22]. Our problem resembles a stochastic control problem. The idea of a so called verification argument is a common method in literature (see e.g. [98]). One takes a solution of a differential equation and shows that under some conditions this coincides with a solution of an optimisation problem. First, we will prove that $P_* = (P_*(\mathtt{e}_1, y), \ldots, P_*(\mathtt{e}_1, y))$ defined in Theorem 12 is a viscosity supersolution.

Lemma 14. *P_* is lower semicontinuous in every entry.*

Proof. Let $(y_k)_{k\geq 0}$ be a sequence converging to y. We define a stopping time

$$\tau_k = \inf\{s \geq t \ : \ D_s = y, D_t = y_k\}. \tag{4.56}$$

Obviously, $\lim_{k\to\infty} \tau_k = t$. Since $P_*(\mathtt{e}_j, y)$ is an equilibrium price, for $\iota = 1, 2$ holds

$$\begin{aligned} \liminf_{k\to\infty} P_*(\mathtt{e}_j, y_k) &\geq \liminf_{k\to\infty} \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^{\tau_k} e^{-\int_t^s \langle \mathtt{r}, X_u\rangle du} D_s ds \middle| X_t = \mathtt{e}_j, D_t = y_k\right) \\ &+ \liminf_{k\to\infty} \mathbb{E}^{\mathbb{P}^\iota}\left(e^{-\int_t^{\tau_k} \langle \mathtt{r}, X_u\rangle du} P_*(X_{\tau_k}, y) \middle| X_t = \mathtt{e}_j, D_t = y_k\right) \end{aligned}$$

By the dominated convergence theorem, the first expectation is zero. Using the independence of X and D and applying again the dominated convergence theorem, we obtain

$$
\begin{aligned}
&\liminf_{k\to\infty} P_*\left(\mathsf{e}_j, y_k\right) \geq \\
&= \mathbb{E}^{\mathbb{P}^\iota}\left(\liminf_{k\to\infty} e^{-\int_t^{\tau_k}\langle \mathtt{r}, X_u\rangle du} \sum_{l=1}^N P_*\left(\mathsf{e}_l, y\right) \mathbf{1}_{X_{\tau_k}=\mathsf{e}_l} \middle| X_t = \mathsf{e}_j, D_t = y_k\right) \\
&= \mathbb{E}^{\mathbb{P}^\iota}\left(\liminf_{k\to\infty} \sum_{l=1}^N P_*\left(\mathsf{e}_j, y\right) \mathbf{1}_{X_{\tau_k}=\mathsf{e}_l} \middle| X_t = \mathsf{e}_j, D_t = y_k\right) \\
&= \sum_{l=1}^N P_*\left(\mathsf{e}_j, y\right) \liminf_{k\to\infty} \mathbb{P}^\iota\left(X_{\tau_k} = \mathsf{e}_l | X_t = \mathsf{e}_j\right) = P_*\left(\mathsf{e}_j, y\right).
\end{aligned}
$$

□

With the help of the last result, we can show the following lemma.

Lemma 15. *P_* is a viscosity supersolution.*

Proof. Suppose P_* is a not viscosity supersolution. Then there exists a vector $\psi(y) = (\psi_1(y), \dots, \psi_N(y)) \in C^2(\mathbb{R}^N)$ and local maximum point $\hat{y}$ of $\psi(y) - P_*(y)$ that satisfies $\psi(\hat{y}) = P_*(\hat{y})$ and

$$
-\max_{\iota\in\{1,2\}}\left(\lambda_\iota\left(\mu-\hat{y}\right)\psi_j'(\hat{y}) + \frac{\sigma^2}{2}\psi_j''(\hat{y}) + \left(A_{jj}^\iota - r_j\right)P_*\left(\mathsf{e}_j,\hat{y}\right) + \hat{y} + f_j^\iota(\hat{y})\right) \leq -\delta \tag{4.57}
$$

for every component $j = 1, \dots, N$ where

$$
f_j^\iota(y) = \sum_{k\neq j} P_*\left(\mathsf{e}_k, y\right) A_{kj}^\iota \tag{4.58}
$$

and $\delta > 0$. Now we show that this leads to a contradiction. For $\epsilon > 0$ let us choose an interval $[\hat{y}-\epsilon, \hat{y}+\epsilon]$ on which $\psi_j(y) - P_*\left(\mathsf{e}_j, y\right) \leq 0$ and

$$
\max_{\iota\in\{1,2\}} \lambda_\iota\left(\mu - y\right)\psi_j'(y) + \frac{\sigma^2}{2}\psi_j''(y) + \left(A_{jj}^\iota - r_j\right)P_*\left(\mathsf{e}_j, y\right) + f_j^\iota(y) \geq \delta \tag{4.59}
$$

for $j = 1, \dots, N$ hold. Let X_t be in state j_0. We define a stopping time

$$
\tau = \inf\left\{s \geq t \ : \ D_s = \hat{y} - \epsilon \ \vee \ D_s = \hat{y} + \epsilon, D_t = \hat{y}, X_t = \mathsf{e}_{j_0}\right\}. \tag{4.60}
$$

Obviously, we can see that $P(\tau > 0) = 1$. Let X be the state in which j_1 when we stop, i.e. $X_\tau = \mathsf{e}_{j_1}$. Proceeding analogously as in the proof of Theorem 13, we apply the Itô formula onto

$$
e^{-\int_t^s \langle \mathtt{r}, X_u\rangle du} \sum_{j=1}^N \psi_j(D_s)\mathbf{1}_{X_t=\mathsf{e}_j}, \tag{4.61}
$$

integrate and omit the terms integrated with respect to martingales. Hence, we receive

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\iota}\left(e^{-\int_t^\tau \langle \mathtt{r}, X_u\rangle du}\psi_{j_1}(D_\tau)ds \middle| D_t = \hat{y}, X_t = \mathsf{e}_{j_0}\right) \\
&= \psi_{j_0}(\hat{y}) + \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\tau e^{-\int_t^{s-}\langle \mathtt{r}, X_u\rangle du}\left(\lambda_\iota\left(\mu - D_s\right)\psi_{j_0}'(D_s) + \frac{\sigma^2}{2}\psi_{j_0}''(D_s)\right.\right. \\
&\quad \left.\left. + \left(A_{j_0j_0}^\iota - r_{j_0}\right)\psi_{j_0}(D_s) + f_{j_0}^\iota(D_s)\right) ds \middle| D_t = \hat{y}, X_t = \mathsf{e}_{j_0}\right)
\end{aligned}
$$

for both investor groups. Since $A^{\iota}_{j_0j_0} - r_{j_0} \leq 0$ and $\psi_{j_0}(y) \leq P_*(\mathbf{e}_{j_0}, y)$, we get

$$\begin{aligned}\max_{\iota\in\{1,2\}} \mathbb{E}^{\mathbb{P}^{\iota}}\left(e^{-\int_t^{\mathcal{T}}\langle \mathbf{r}, X_u\rangle du}\psi_{j_1}(D_{\mathcal{T}})\Big| D_t = \hat{y}, X_t = \mathbf{e}_{j_0}\right) \geq \psi_{j_0}(\hat{y})\\
+\max_{\iota\in\{1,2\}} \mathbb{E}^{\mathbb{P}^{\iota}}\Bigg(\int_t^{\mathcal{T}} e^{-\int_t^s \langle \mathbf{r}, X_u\rangle du}\left(\lambda_\iota(\mu - D_s)\psi'_{j_0}(D_s) + \frac{\sigma^2}{2}\psi''_{j_0}(D_s)\right.\\
\left. + \left(A^{\iota}_{j_0j_0} - r_{j_0}\right)P_*(\mathbf{e}_{j_0}, D_s) + f^{\iota}_{j_0}(D_s)\right) ds \Big| D_t = \hat{y}, X_t = \mathbf{e}_{j_0}\Bigg)\\
\geq \psi_{j_0}(\hat{y}) + \max_{\iota\in\{1,2\}} \mathbb{E}^{\mathbb{P}^{\iota}}\left(\int_t^{\mathcal{T}} e^{-\int_t^s \langle \mathbf{r}, X_u\rangle du}\delta ds\Big| D_t = \hat{y}, X_t = \mathbf{e}_{j_0}\right).\end{aligned}$$

Due to Lemma 11 we obtain

$$\max_{\iota\in\{1,2\}} \mathbb{E}^{\mathbb{P}^{\iota}}\left(e^{-\int_t^{\mathcal{T}}\langle \mathbf{r}, X_u\rangle du}\psi_{j_1}(D_{\mathcal{T}})\Big| D_t = \hat{y}, X_t = \mathbf{e}_j\right) \geq \psi_{j_0}(\hat{y}) + \delta \max_{\iota\in\{1,2\}} c_\iota \tag{4.62}$$

where

$$c_\iota = \max_{\iota\in\{1,2\}} \int_t^{\mathcal{T}} \left\langle e^{(s-t)(\mathbf{A}^{\iota} - \mathrm{diag}(\mathbf{r}))}\mathbf{e}_{j_0}\left(\mu + e^{-\lambda_\iota(s-t)}(\hat{y} - \mu)\right), \mathbf{1}\right\rangle ds > 0. \tag{4.63}$$

We also know that $P_*(y)$ is an equilibrium price and therefore

$$\begin{aligned}&p^*_{j_0}(\hat{y}) \geq\\
&\max_{\iota\in\{1,2\}} \mathbb{E}^{\mathbb{P}^{\iota}}\left(\int_t^{\mathcal{T}} e^{-\int_t^s\langle \mathbf{r}, X_u\rangle du} D_s ds + e^{-\int_t^{\mathcal{T}}\langle \mathbf{r}, X_u\rangle du} P_*(\mathbf{e}_{j_1}, y)\Big| X_t = \mathbf{e}_j, D_t = \hat{y}\right).\end{aligned} \tag{4.64}$$

Combining (4.62) and (4.64) with $P_*(\mathbf{e}_{j_1}, y) - \psi_{j_1}(y) \geq 0$, we obtain

$$\begin{aligned}0 &= P_*(\mathbf{e}_{j_0}, \hat{y}) - \psi_{j_0}(\hat{y})\\
&\geq \max_{\iota\in\{1,2\}} \mathbb{E}^{\mathbb{P}^{\iota}}\left(e^{-\int_t^{\mathcal{T}}\langle \mathbf{r}, X_u\rangle du}\left(P_*(\mathbf{e}_{j_1}, D_{\mathcal{T}}) - \psi_{j_1}(D_{\mathcal{T}})\right) ds\Big| D_t = \hat{y}, X_t = \mathbf{e}_j\right) + \delta c_\iota\\
&\geq \max_{\iota\in\{1,2\}} \delta c_\iota > 0,\end{aligned}$$

which is the desired contradiction. Since $P_*(\mathbf{e}_j, y)$ is also lower semicontinuous by Lemma 14, it is a viscosity supersolution. □

Finally, the following theorem gives us a series representation of the minimal equilibrium price. In the proof we need the at most linear growth of an equilibrium price. This is always guaranteed when both investors agree on the rate matrix for the regime switching by Theorem 14 where we even get an explicit solution of the associated differential equation for the equilibrium price.

Theorem 15. *The unique solution of system of the differential equations with linear growth at infinity is the minimal equilibrium price, i.e.* $\Phi(y) = P_*(y)$.

Proof. Since $P_*(y)$ is the minimal equilibrium price and $\Phi(y)$ is an equilibrium price, $P_*(y) \leq \Phi(y)$ obviously holds. Proceeding state wise analogously to [22], we distinguish into two cases. Let

$$\inf_{y\in\mathbb{R}} \left(P_*(y) - \Phi(y)\right) \tag{4.65}$$

be unbounded. Due to $\Phi(y) - P_*(y) \leq \Phi(y) - I(y)$, we see that

$$\lim_{|y|\to\infty} \left(P_*(y) - \Phi(y)\right) = 0. \tag{4.66}$$

In the other case, when

$$\inf_{y\in\mathbb{R}} \left(P_*(y) - \Phi(y)\right) \tag{4.67}$$

is bounded, there exists $\hat{y} \in \mathbb{R}$, where the minimum is achieved. Since $P_*(y)$ is a viscosity supersolution due to Lemma 15,

$$-\max_{\iota\in\{1,2\}} \left(\lambda_\iota \left(\mu - \hat{y}\right) \Phi_j'(\hat{y}) + \frac{\sigma^2}{2}\Phi_j''(\hat{y}) + K_{jj}^\iota P_* \left(\mathtt{e}_j, \hat{y}\right) + \hat{y} + f_j^\iota(\hat{y})\right) \geq 0 \tag{4.68}$$

for every component $j = 1, \ldots, N$ where

$$f_j^\iota(y) = \sum_{k\neq j} P_* \left(\mathtt{e}_k, y\right) A_{kj}^\iota. \tag{4.69}$$

On the other hand $\Phi(y)$ is also a solution of the differential equation (4.49). Hence, we get

$$-\max_{\iota\in\{1,2\}} \left(K_{jj}^\iota P_* \left(\mathtt{e}_j, \hat{y}\right) - K_{jj}^\iota \Phi_j \left(\hat{y}\right)\right) \geq 0. \tag{4.70}$$

As $K_{jj}^\iota = A_{jj}^\iota - r_j < 0$ and $\hat{y}$ is the minimal point of $P_* \left(\mathtt{e}_j, y\right) - \Phi_j \left(y\right)$, we obtain

$$P_* \left(\mathtt{e}_j, y\right) - \Phi_j \left(y\right) \geq P_* \left(\mathtt{e}_j, \hat{y}\right) - \Phi_j \left(\hat{y}\right) \geq 0 \tag{4.71}$$

for every j and therefore $P_*(y) - \Phi(y) \geq 0$. □

4.4. Asset Bubbles: Results, Numerical Examples and Discussion

We define a bubble as

$$B(\mathtt{x}, y) = \mathtt{x}^\mathsf{T} \left(\Phi(y) - I(y)\right),$$

i. e. as the difference between minimal equilibrium price and intrinsic value (as in any other equilibrium model like [21] or [89]). Finally, for investor groups agreeing on the rate matrix for the regime switching, Theorem 14 and Theorem 15 provide us an explicit representation of an asset bubble as

$$B(\mathtt{x}, y) = \begin{cases} \mathtt{x}^\mathsf{T} \left(\mathrm{F}_{\frac{\mathtt{K}\mathsf{T}}{\lambda_2}} \left(\frac{\sqrt{2\lambda_2}}{\sigma}(y-\mu)\right) \mathtt{k}_2 + \tilde{I}_2(y) - I_2(y)\right) & \text{for } y \leq \mu, \\ \mathtt{x}^\mathsf{T} \left(\mathrm{F}_{\frac{\mathtt{K}\mathsf{T}}{\lambda_1}} \left(\frac{\sqrt{2\lambda_1}}{\sigma}(\mu-y)\right) \mathtt{k}_1 + \tilde{I}_1(y) - I_1(y)\right) & \text{for } y > \mu. \end{cases} \tag{4.72}$$

We start with a simple example with three different states of the interest rate. We choose the generator matrix

$$A^\mathsf{T} = \begin{pmatrix} -2 & 1.5 & 0.5 \\ 0 & -0.25 & 0.25 \\ 0.05 & 0.9 & -0.95 \end{pmatrix}, \tag{4.73}$$

and the vector of interest rate

$$R = \begin{pmatrix} 0.002 \\ 0.008 \\ 0.01 \end{pmatrix} \tag{4.74}$$

the other model parameters are set

$$\lambda_1 = 0.2, \lambda_2 = 0.4, \mu = 0.24, \sigma = 0.05. \tag{4.75}$$

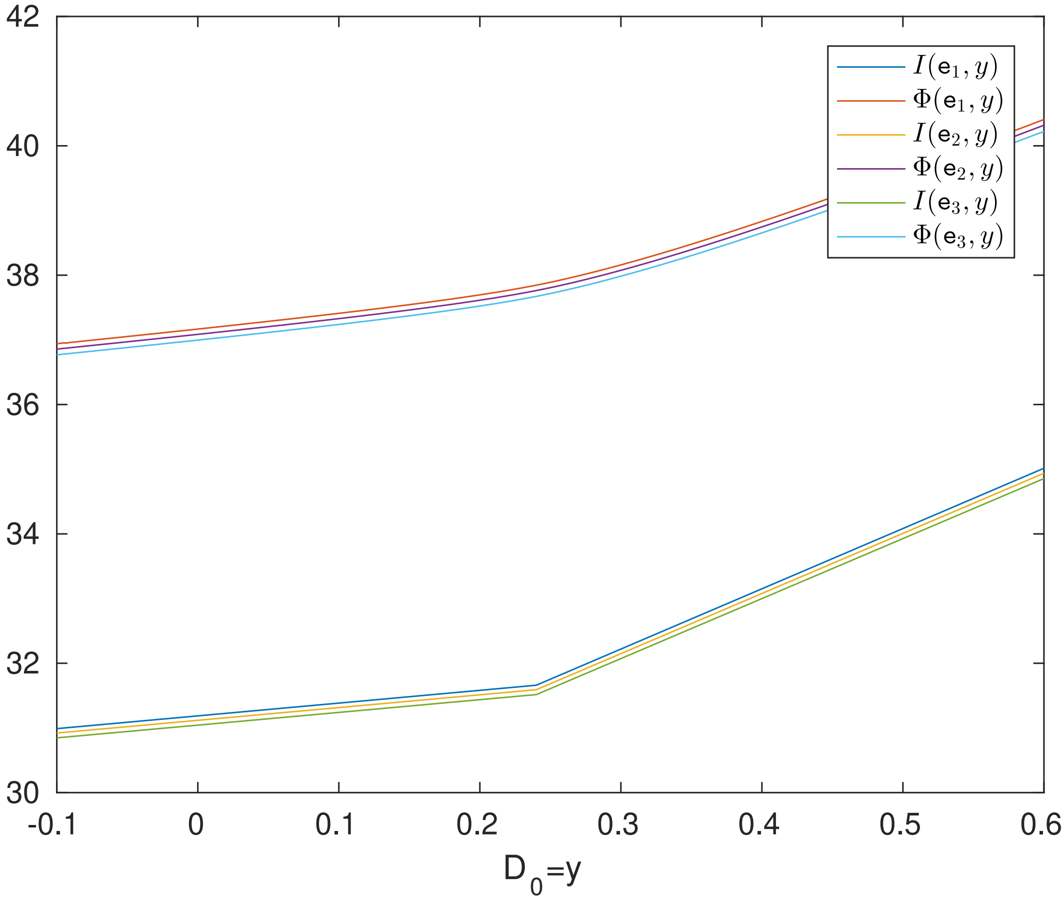

Figure 4.1.: Mininmal equilibrium price and intrinsic value.

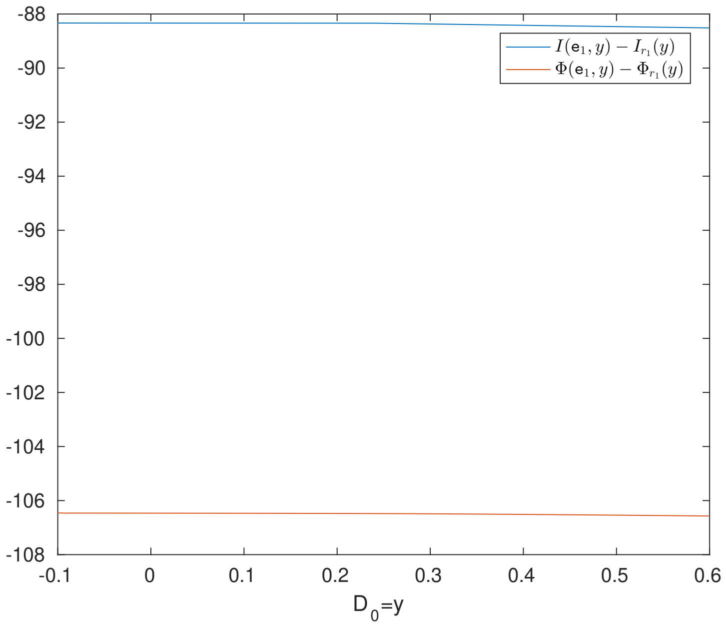

Figure 4.2.: Difference between mininmal equilibrium prices and intrinsic values with regime switching and with a fixed interest rate $r_1 = 0.002$.

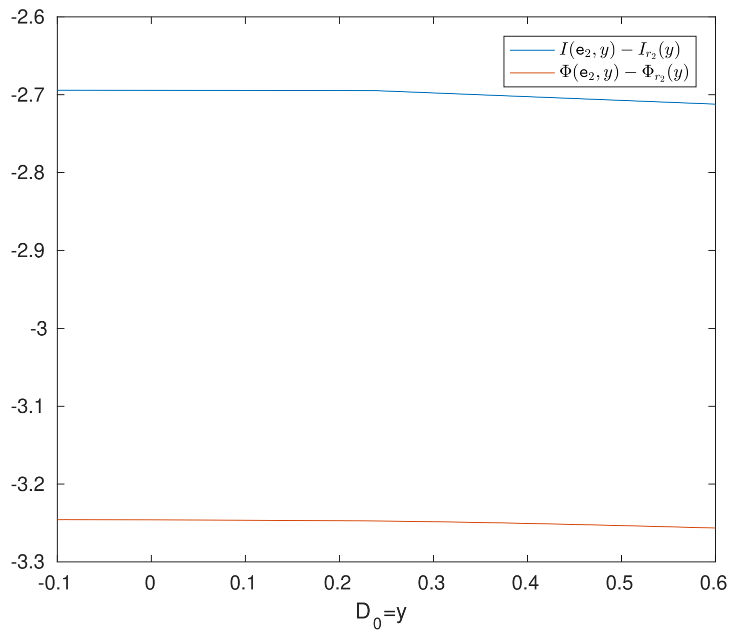

Figure 4.3.: Difference between mininmal equilibrium prices and intrinsic values with regime switching and with a fixed interest rate $r_2 = 0.008$.

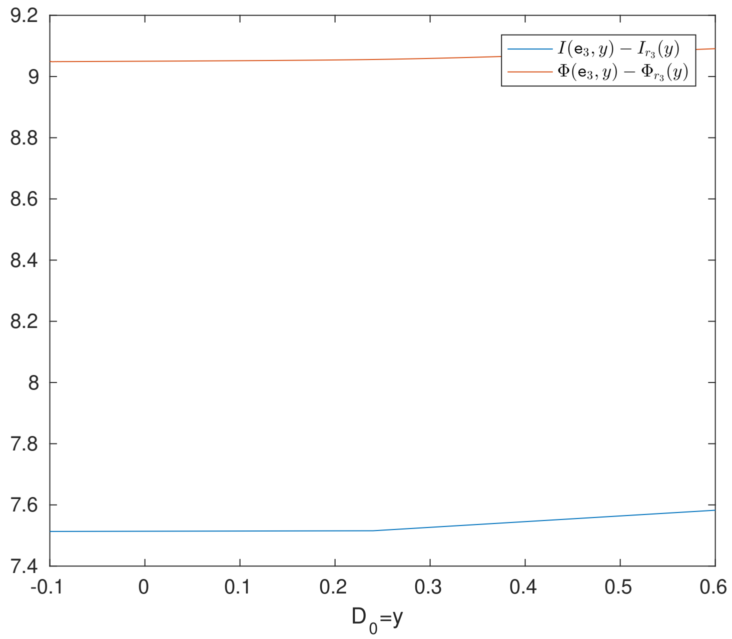

Figure 4.4.: Difference between mininmal equilibrium prices and intrinsic values with regime switching and with a fixed interest rate $r_3 = 0.01$.

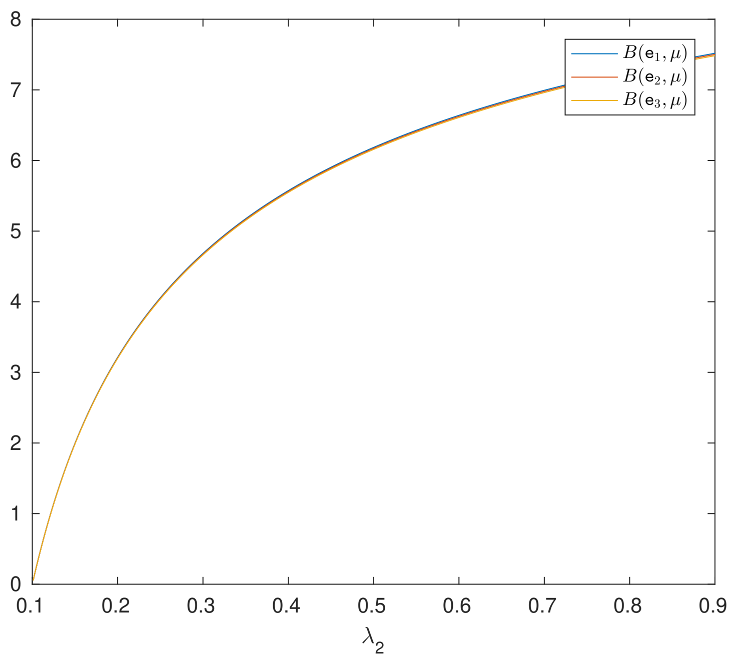

Figure 4.5.: Bubbles size for $D_0 = \mu$ depending on λ_2.

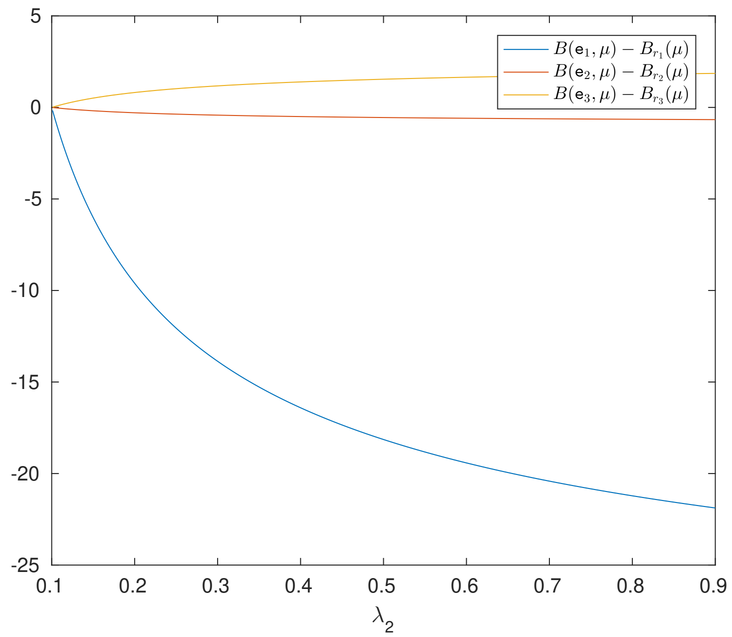

Figure 4.6.: Difference between bubbles with regime switching and with fixed interest rates $r_1 = 0.002, r_2 = 0.008, r_3 = 0.01$ for $D_0 = \mu$ depending on λ_2.

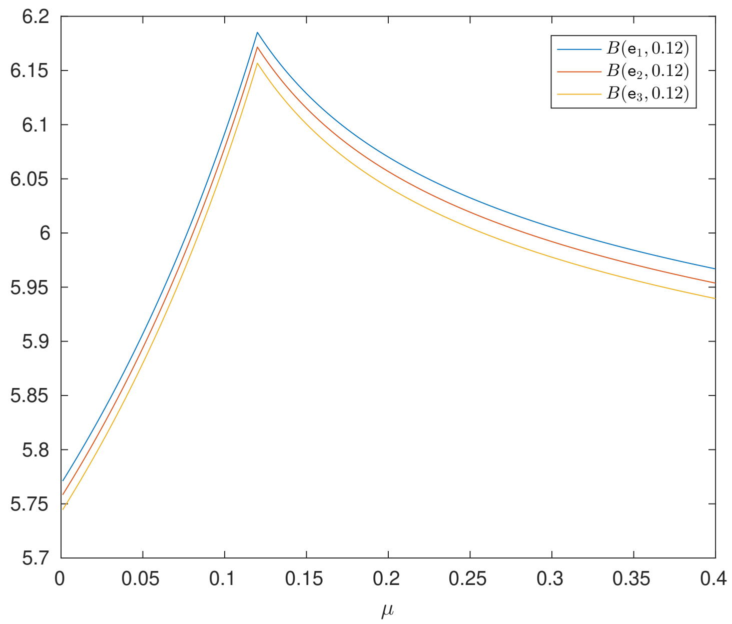

Figure 4.7.: Bubbles size for $D_0 = 0.12$ depending on μ.

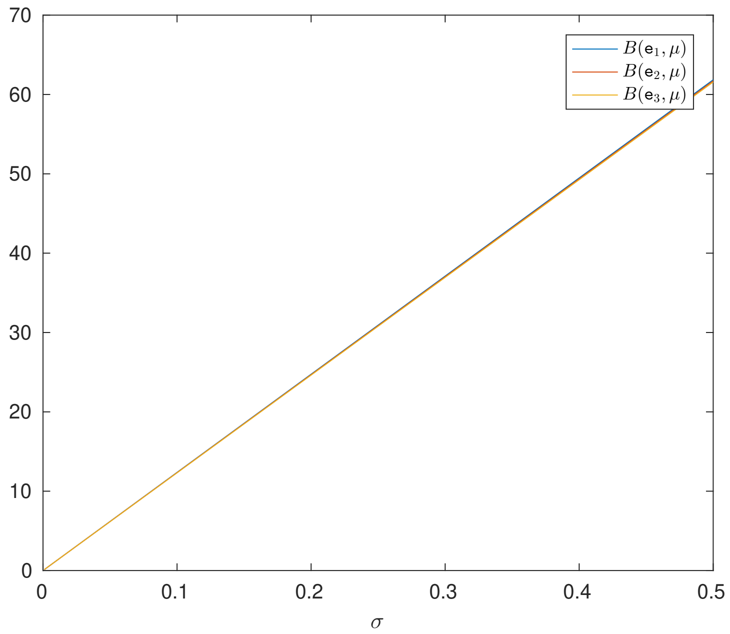

Figure 4.8.: Bubbles size with for $D_0 = \mu$ depending on σ.

Figure 4.1 depicts the minimal equilibrium price and intrinsic value depending on D_0 for each X_0. Since $r_1 < r_2 < r_3$, we can see in both that the higher the interest rate is in the initial state, the lower they get. Obviously, for the same expected dividend scenario, investors are willing to pay less under a better interest rate situation. A quite remarkable property of the regime switching model is shown by Figure 4.2. The difference between a classical model with r_1 as fix interest rate and the switching model is enormous. Without the possibility to switch to another interest rate state, the prices and their difference, hence also the bubble, get at least three times larger. To understand this, we take a look at the probabilities to stay in state e_1. In one time unit, the probability to leave this state is 0.86. This gets even clearer, if we compute the stationary distribution $(0.005236, 0.785340, 0.209424)^\intercal$. Therefore, it obvious to be very likely to get a better interest rate than r_1. Figure 4.3 shows a similar behaviour, because it is more likely to switch to a better interest rate than to a worse. Figure 4.4 compares a model with $r_3 = 0.01$ to the regime switching model starting in $X_0 = \mathsf{e}_3$. Of course, in this case, intrinsic value and minimal equilibrium price are lower in the simple model, because the interest rate cannot get lower. Many monotonicity properties from [21] also hold in more general regime switching model. Figure 4.5 shows that bubbles are increasing in $\lambda_2 - \lambda_1$. The more heterogeneously markets participants anticipate the dividend process, the larger the bubbles get. Moreover, Figure 4.6 shows that under possibility to switch to a better interest rate, the difference between the bubble with and without regime switch gets larger for growing λ_2. In Figure 4.7, we can see that the bubble is smaller, if the process D starts far away from μ. In that case, one participant is clearly more optimistic than the other, which influences the price. Finally, Figure 4.8 depicts an almost linear growth of the bubble in σ.

Let us now consider a two state economy with one absorbing state. We choose the parameters as:

$$
\begin{aligned}
A^{\intercal} &= \begin{pmatrix} -0.5 & 0.5 \\ 0 & 0 \end{pmatrix}, \\
R &= \begin{pmatrix} 0.0075 \\ 0.021 \end{pmatrix}, \\
\lambda_1 &= 0.1, \\
\lambda_2 &= 0.4, \\
\mu &= 0.24, \\
\sigma &= 0.02.
\end{aligned}
$$

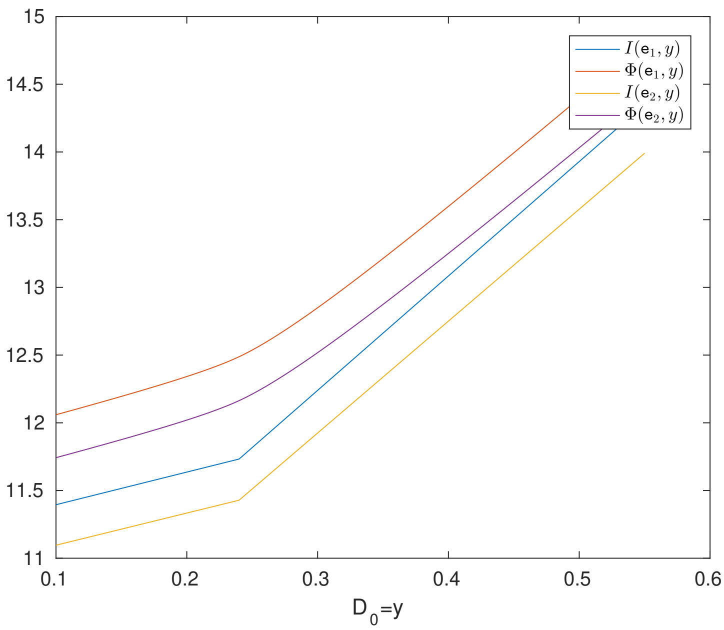

Figure 4.9.: Mininmal equilibrium price and intrinsic value.

Sometimes, numerical problems arise in inverting $\mathtt{K} = \mathtt{A} + \mathrm{diag}(\mathbf{r})$. Since the column sum of $\mathtt{A}$ is zero, $\mathtt{1}$ is an eigenvector to the eigenvalue 0 and hence $\mathtt{A}$ is always singular. Although $\mathtt{K}$ is non-singular, its determinant can become very small according to the choice of of the interest vector $\mathrm{diag}(\mathbf{r})$. An interesting point is also how to implement the matrix gamma function $\Gamma(\mathtt{M})$ numerically. In our case we restrict ourselves to diagonisable matrices. Then we decompose

$$\Gamma(\mathtt{M}) = \Gamma(\mathtt{TDT}^{-1}) = \int_0^\infty e^{-t}\mathtt{T}e^{\mathtt{D}\ln(t)}\mathtt{T}^{-1}t^{-1}dt = \mathtt{T}\Gamma(\mathtt{G})\mathtt{T}^{-1} \tag{4.76}$$

where the matrix

$$\Gamma(\mathtt{G}) = \begin{pmatrix} \Gamma(\mu_1) & 0 & \dots & 0 \\ 0 & \Gamma(\mu_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Gamma(\mu_N) \end{pmatrix} \tag{4.77}$$

can be calculated using common numerical methods for the gamma functions of the eigenvalues $\mu_1, \dots, \mu_N$ of $\mathtt{M}$ in its diagonal entries. Another approach is to compute the

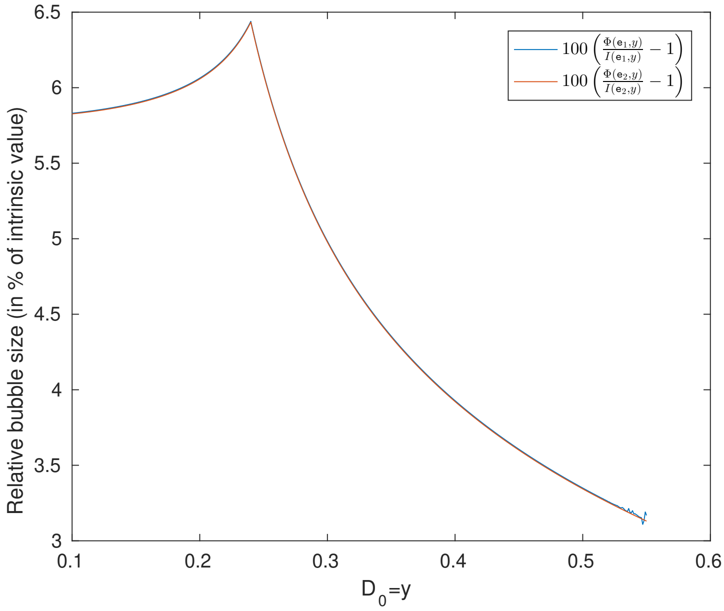

Figure 4.10.: Relative bubble size.

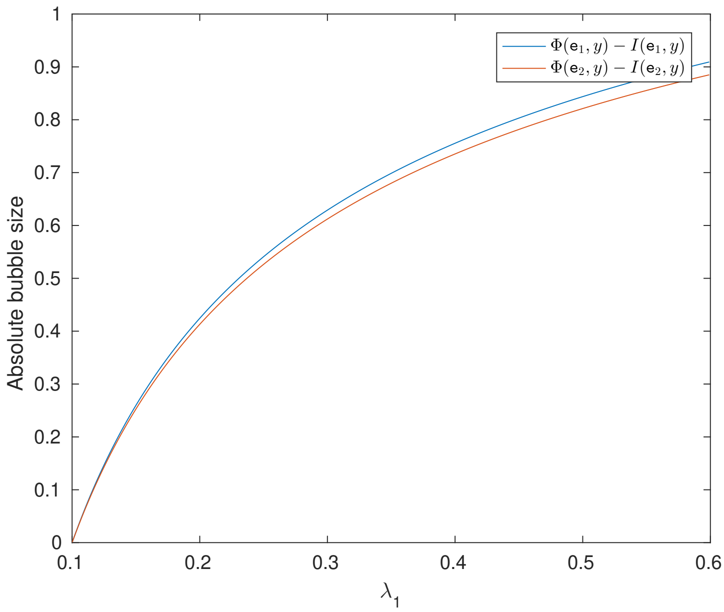

Figure 4.11.: Bubble size depending on λ_1.

integral

$$\Gamma(\mathtt{M}) = \int_0^\infty e^{(\mathtt{M}-\mathtt{I})\ln(t)} e^{-t} dt \tag{4.78}$$

with the help of a Gauss-Laguerre quadrature (see [1, p. 890]) as

$$\Gamma(\mathtt{M}) \approx \sum_{k=1}^{n} e^{(\mathtt{M}-\mathtt{I})\ln(t)} w_k \tag{4.79}$$

with the weights

$$w_k = \frac{x_k^n}{(n+1)^2 (L_{n+1}(x_k^n))^2} \tag{4.80}$$

where L_n is the n-th Laguerre polynomial (see [1, p. 778]) and x_k^n is the k-th root of it. The drawback of this method is that the integral definition of the gamma matrix function only holds for positive stable matrices. The classical definition of the gamma function due to Weierstrass can be adapted for the gamma matrix function (see [65]), but since we could not find an error estimate, this is useless for numerical implementation.

5. A Lévy Model for Asset Bubbles

5.1. Model Setting

In this chapter we set up a model for asset bubbles in an arbitrage free market with just one risky asset. Since this model generalises the idea of Chen and Kohn [21] to Lévy processes, we have to start with several technical assumptions. Let $L^1 = \left(L_t^1\right)_{t\geq 0}$ be a one-dimensional, càdlàg Lévy process with characteristic triplet (a, ν, γ_1) on a filtered probability space $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^1\right)$ satisfying the usual hypothesis. We exclude pure jump processes such that $a \neq 0$. The market has two investor groups that have no other investment possibilities than the risky asset. So, they compete only under each other. We begin with the first group. Let the asset's dividend rate process $D = (D_t)_{t\geq 0}$ be defined via the Lévy driven Ornstein-Uhlenbeck equation

$$dD_t = \lambda_1(\mu - D_{t-})dt + \sigma dL_t^1 \tag{5.1}$$

with initial value $D_0 = x_0 \in \mathbb{R}$. The parameters are chosen $\mu \geq 0$, $\sigma > 0$ and $\lambda_1 > 0$. From Lemma 2, we know the unique solution

$$D_t = \mu + e^{-\lambda_1 t}(x - \mu) + \sigma \int_0^t e^{-\lambda_1(t-s)} dL_s^1. \tag{5.2}$$

As discussed in Section 2.1, D is a stationary process and in the case $x = 0$ even a Lévy process. The second group considers a similar development, but under another probability measure. Therefore, we introduce another process L_t^2 through

$$dL_t^2 = dL_t^1 - \frac{\lambda_2 - \lambda_1}{\sigma}\left(\mu - D_{t-}\right) dt \tag{5.3}$$

where $\lambda_2 > \lambda_1$. Our aim is now to find a measure $\mathbb{P}^2$ under which L^2 is also a Lévy process. Using the Lévy- Itô decomposition (see Theorem 2),

$$L_t^1 = \gamma_1 t + aW_t^1 + \int_0^t \int_{|z|\leq 1} z\left(J(dz, ds) - \nu(dz)ds\right) + \int_0^t \int_{|z|>1} zJ(dz, ds), \tag{5.4}$$

and combining it with the definition of L_t^2, we obtain

$$\begin{aligned} L_t^2 &= L_t^1 - \int_0^t \frac{\lambda_2 - \lambda_1}{\sigma}\left(\mu - D_{s-}\right) ds \\ &= \gamma_1 t + \left(aW_t^1 + \int_0^t \frac{\lambda_1 - \lambda_2}{\sigma}\left(\mu - D_{s-}\right) ds\right) \\ &\quad + \int_0^t \int_{|z|\leq 1} z\left(J(dz, ds) - \nu(dz)ds\right) + \int_0^t \int_{|z|>1} zJ(dz, ds). \end{aligned}$$

Now we define

$$W_t^2 = W_t^1 + \frac{1}{a}\int_0^t \frac{\lambda_1 - \lambda_2}{\sigma}\left(\mu - D_{s-}\right) ds. \tag{5.5}$$

Further we assume that the integrand satisfies the Novikov condition (see Theorem 6),

$$\mathbb{E}^{\mathbb{P}^1}\left(\exp\left(\frac{1}{2}\int_0^T\left(\frac{\lambda_2-\lambda_1}{\sigma a}\right)^2(\mu-D_{s-})^2\,dt\right)\right)<\infty, \tag{5.6}$$

for all $0\leq T<\infty$. Therefore, we can apply Girsanov's theorem (see Theorem 7). Hence, W_t^2 is a Brownian motion with respect to $\mathbb{P}^2$ and therefore, L_t^2 is Lévy process under the measure $\mathbb{P}^2$ with the characteristic triplet (a,ν,γ_2).

Let us now consider trading in this model. We assume that the market participants can lent or borrow money at a fixed interest rate $r>0$ and that they are always liquid. Let short selling be forbidden. Further, we suppose that there are constant transaction costs in the following sense: We impose on the seller that for each transaction he has to give away κ_p percent of the price to a third party, e.g. a governmental institution. For simplicity reasons, we shall write from now on $\kappa=1-\kappa_p/100$. Assuming $0\leq\kappa_p\leq 100$, we know that $0\leq\kappa\leq 1$. Let us assume for a moment a market price at time t as an $\mathcal{F}_t$-measurable random variable. We consider only two classes of market participants who both act risk neutral, i.e. they maximise the expected linear utility of their wealth. As in the model by Chen and Kohn [21], the group ι holds u_t^ι shares of the asset at time t. Hence, each investor group maximises

$$\mathbb{E}^{\mathbb{P}^\iota}\left(u_t^\iota\sup_{\tau\geq t}\left(\int_t^\tau e^{-r(s-t)}D_sds+\kappa e^{-r(\tau-t)}p_\tau\right)+\left(1-u_t^\iota\right)p_t\right) \tag{5.7}$$

in order to choose the portfolio. The short selling constraint implies $0\leq u_t^\iota\leq 1$. Therefore, we obtain $u_t^\iota=0$ or $u_t^\iota=1$ and so the asset is at any time held by the more optimistic group.

5.2. Intrinsic Value and Equilibrium Price

The *intrinsic value* at time $t\geq 0$ is defined as the maximal price, an investor is willing to pay for all expected discounted future dividends assuming he cannot resell (see [21,99]), i.e.

$$I(x,t)=\max_{\iota=1,2}\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\infty e^{-r(s-t)}D_sds\middle|D_t=x\right). \tag{5.8}$$

Let us further assume that the integral above exists, which is true for the special case we see in the following lemma.

Lemma 16. *For $\alpha\in(0,1)$ et L^ι be a Lévy process with α-stable jumps. For $\alpha\in[1,2]$ let L^ι be a Lévy process with symmetric-α-stable jumps. Then the intrinsic value is time-independent and can be written as*

$$I(x)=I\left(x,t\right)=\begin{cases}\frac{x}{r+\lambda_2}+\frac{\mu\lambda_2+\sigma\left(\gamma_2+\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(0,1)}\right)}{r(r+\lambda_2)} & \text{for } x<x_I,\\ \frac{x}{r+\lambda_1}+\frac{\mu\lambda_1+\sigma\left(\gamma_1+\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(0,1)}\right)}{r(r+\lambda_1)} & \text{for } x\geq x_I,\end{cases} \tag{5.9}$$

where

$$x_I=\mu+\frac{\sigma}{r}\left(\frac{\lambda_1\gamma_2-\lambda_2\gamma_1+r\left(\gamma_2-\gamma_1\right)}{\left(\lambda_2-\lambda_1\right)}-\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(0,1)}\right). \tag{5.10}$$

Proof. Using a stochastic version of Fubini's theorem (see [83, p. 207f]), we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\infty e^{-r(s-t)} D_s ds \middle| D_t = x\right) \\
&= \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\infty \left(\mu + e^{-\lambda_\iota(s-t)}(D_t - \mu)\right) e^{-r(s-t)} ds \right. \\
&\quad \left. + \sigma \int_t^\infty \int_t^s e^{-\lambda_\iota(s-u)} e^{-r(s-t)} dL_u^\iota ds \middle| D_t = x\right) \\
&= \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \sigma e^{rt} \mathbb{E}^{\mathbb{P}^\iota}\left(\lim_{\tau\to\infty} \int_t^\tau \int_t^s e^{-(\lambda_\iota + r)s + \lambda_\iota u} dL_u^\iota ds \middle| D_t = x\right) \\
&= \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \sigma e^{rt} \mathbb{E}^{\mathbb{P}^\iota}\left(\lim_{\tau\to\infty} \int_t^\tau \left(\int_u^\tau e^{-(\lambda_\iota + r)s} ds\right) e^{\lambda_\iota u} dL_u^\iota \middle| D_t = x\right) \\
&= \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \sigma e^{rt} \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\infty \frac{e^{-(\lambda_\iota + r)u}}{\lambda_\iota + r} e^{\lambda_\iota u} dL_u^\iota \middle| D_t = x\right) \\
&= \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \frac{\sigma e^{rt}}{\lambda_\iota + r} \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\infty e^{-ru} dL_u^\iota \middle| D_t = x\right).
\end{aligned}$$

Our next steps is to decompose L^ι in order to compute the integral expression above. We shall approximate the jump part through the limit of a compound Poisson process. Let $0 < \varepsilon \leq 1$. Let us define a compound Poisson process

$$X_t^\varepsilon = \int_0^t \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} z J(dz, ds) \tag{5.11}$$

with the Poisson random measure J having the intensity $\nu(dx)dt$. In other words, this process only comprises the jumps of L^ι with absolute value larger than ε. Note that by construction, both investor groups agree on the jumps of L^ι which means that $\Delta L_t^1 = \Delta L_t^2$ for all $t \geq 0$. Hence, we can also write

$$X_t^\varepsilon = \sum_{\substack{0\leq s\leq t \\ |\Delta L_s^\iota|\geq\varepsilon}} \Delta L_s^\iota. \tag{5.12}$$

For $\varepsilon > 0$, this sum converges, but ν can have a singularity at zero. Therefore, we need to compensate the small jumps as in the Lévy- Itô decomposition (see Theorem 2). We start with the case $\alpha \in (0, 1)$. The compensator has the form

$$A_t^\varepsilon = t \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} z\nu(dz). \tag{5.13}$$

Using the notation from section 2.2, we get

$$\begin{aligned}
A_t^\varepsilon &= t\left(c_2 \int_{-1}^{-\varepsilon} \frac{z}{(-z)^{\alpha+1}} dz + c_1 \int_\varepsilon^1 \frac{z}{z^{\alpha+1}} dz\right) \\
&= t\left(c_2 \frac{1-\varepsilon^{1-\alpha}}{\alpha - 1} - c_1 \frac{1-\varepsilon^{1-\alpha}}{\alpha-1}\right) = t\,(c_2 - c_1)\,\frac{1-\varepsilon^{1-\alpha}}{\alpha-1}
\end{aligned}$$

and therefore

$$\lim_{\varepsilon\to 0^+} A_t^\varepsilon = t\frac{c_2 - c_1}{\alpha - 1}. \tag{5.14}$$

Obviously, the limit above does not converge for $\alpha \geq 1$. For $\alpha \in [1, 2]$, we need to assume additionally symmetry, i.e. $c_1 = c_2$. In that case, we receive $A_t^\varepsilon = 0$. Later, we will also use this restriction. Now we examine the parameter λ_ε of the compound Poisson process X^ε. We can compute

$$\lambda_\varepsilon = \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \nu(dz) = \int_{-\infty}^{-\varepsilon} \frac{c_2}{(-z)^{\alpha+1}} dz + \int_{\varepsilon}^{\infty} \frac{c_1}{z^{\alpha+1}} dz = \frac{c_1+c_2}{\alpha\varepsilon^\alpha}. \tag{5.15}$$

Let $(N_t^\varepsilon)_{t\geq 0}$ be a Poisson process with rate λ_ε. We define a new measure

$$\nu_\varepsilon(A) = \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \mathbf{1}_A(z)\nu(dz) \tag{5.16}$$

for $A \in \mathfrak{B}(\mathbb{R})$. Obviously, $\nu_\varepsilon(\mathbb{R}) = \lambda_\varepsilon$. Let $(Y_k)_{k\in\mathbb{N}_0}$ be a sequence of i.i.d. random variables with distribution $\frac{\nu_\varepsilon(dx)}{\nu_\varepsilon(\mathbb{R})}$ independent of N^ε. Hence, we can write

$$X_t^\varepsilon = \sum_{k=1}^{N_t^\varepsilon} Y_k. \tag{5.17}$$

Let $T_0 = 0$ and denote with $(T_k^\varepsilon)_{k=1,\dots,N_T^\varepsilon}$ the jump times of the Poisson process N^ε on the interval $[0,T]$, then we know that

$$T_k^\varepsilon - T_{k-1}^\varepsilon \overset{\text{i.i.d.}}{\sim} \operatorname{Exp}(\lambda_\varepsilon) \tag{5.18}$$

for every $k = 1,\dots,N_T^\varepsilon$. Suppose we knew the state of N^ε at time t and let $j \in \mathbb{N}_0$ be the amount of jumps until t. With that, we can calculate

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^T e^{-rt} dX_t^\varepsilon \middle| N_t^\varepsilon = j\right) &= \mathbb{E}^{\mathbb{P}^\iota}\left(\sum_{k=N_t^\varepsilon}^{N_T^\varepsilon} e^{-r\left(T_k^\varepsilon - T_{k-1}^\varepsilon\right)} Y_i \middle| N_t^\varepsilon = j\right) \\
&= \sum_{k=j}^{\infty} \mathbb{P}^\iota\left(N_T^\varepsilon = k\right) \mathbb{E}^{\mathbb{P}^\iota}\left(e^{-r\left(T_k^\varepsilon - T_{k-1}^\varepsilon\right)}\right) \mathbb{E}^{\mathbb{P}^\iota}\left(Y_k\right) \\
&= \sum_{k=j}^{\infty} \frac{(\lambda_\varepsilon T)^k}{k!} e^{-\lambda_\varepsilon T} \frac{\lambda_\varepsilon}{\lambda_\varepsilon + r} \mathbb{E}^{\mathbb{P}^\iota}\left(Y_1\right).
\end{aligned}$$

Now we distinguish into two cases: First, let $\alpha \in (0,1)$. Knowing the distribution of Y_1, we get

$$\mathbb{E}^{\mathbb{P}^\iota}\left(Y_1\right) = \frac{1}{\lambda_\varepsilon} \int_{\mathbb{R}} z\nu_\varepsilon(dz) = \frac{1}{\lambda_\varepsilon} \frac{c_1 - c_2}{(\alpha-1)\varepsilon^{\alpha-1}} \tag{5.19}$$

and hence

$$\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^T e^{-rt} dX_t^\varepsilon \middle| N_t^\varepsilon = j\right) = \sum_{k=j}^{\infty} \frac{(\lambda_\varepsilon T)^k}{k!} e^{-\lambda_\varepsilon T} \frac{(c_1 - c_2)\,\alpha\varepsilon}{(c_1 + c_2 + r\varepsilon^\alpha)(\alpha - 1)}. \tag{5.20}$$

Since the limit above holds for every $j \in \mathbb{N}_0$, it follows that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\infty e^{-rt} dX_t^\varepsilon \middle| D_t = x\right) &= \lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^T e^{-rt} dX_t^\varepsilon \middle| D_t = x\right) \\
&= \lim_{T\to\infty} \frac{\left(e^{\lambda_\varepsilon T} - \sum_{k=0}^{j} \frac{(\lambda_\varepsilon T)^k}{(k)!}\right) e^{-\lambda_\varepsilon T} (c_1 - c_2)\,\alpha\varepsilon}{(c_1 + c_2 + r\varepsilon^\alpha)(\alpha - 1)} \\
&= \frac{(c_1 - c_2)\,\alpha\varepsilon}{(c_1 + c_2 + r\varepsilon^\alpha)(\alpha - 1)} \overset{\varepsilon\to 0}{\to} 0.
\end{aligned}$$

By the Lévy- Itô decomposition (see Theorem 2) we get

$$L_t^\iota = \gamma_\iota t + aW_t^\iota + \lim_{\varepsilon \to 0^+} X_t^\varepsilon + A_t^\varepsilon. \tag{5.21}$$

and hence

$$dL_t^\iota = \gamma_\iota dt + adW_t^\iota + \lim_{\varepsilon \to 0^+} dX_t^\varepsilon + \frac{c_2 - c_1}{\alpha - 1} dt. \tag{5.22}$$

Putting everything together, we can now rewrite

$$\begin{aligned} I(x,t) &= \max_{\iota=1,2} \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \frac{\sigma e^{rt}}{\lambda_\iota + r} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\infty e^{-ru} dL_u^\iota \middle| D_t = x \right) \\ &= \max_{\iota=1,2} \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \frac{\sigma e^{rt}}{\lambda_\iota + r} \int_t^\infty e^{-ru} \left(\gamma_\iota + \frac{c_2 - c_1}{\alpha - 1} \right) du \\ &= \max_{\iota=1,2} \frac{\mu}{r} + \frac{xr - \mu r + \sigma \gamma_\iota + \sigma \frac{c_2 - c_1}{\alpha - 1}}{(\lambda_\iota + r)\, r}. \end{aligned}$$

Remembering the fact $\lambda_2 > \lambda_1$, we can solve the inequality

$$\frac{xr - \mu r + \sigma \gamma_1 + \sigma \frac{c_2 - c_1}{\alpha - 1}}{\lambda_1 + r} > \frac{xr - \mu r + \sigma \gamma_2 + \sigma \frac{c_2 - c_1}{\alpha - 1}}{\lambda_2 + r} \tag{5.23}$$

with respect to x and hence find the maximum.

For the case $\alpha \in [1, 2]$ we additionally assume symmetry and hence obtain obviously $\mathbb{E}^{\mathbb{P}^\iota}(Y_1) = 0$. Therefore, we see that

$$\mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\infty e^{-rt} dX_t^\varepsilon \middle| D_t = x \right) = 0 \tag{5.24}$$

and by similar argumentation as above, we get

$$\begin{aligned} I(x,t) &= \max_{\iota=1,2} \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \frac{\sigma e^{rt}}{\lambda_\iota + r} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\infty e^{-ru} dL_u^\iota \middle| D_t = x \right) \\ &= \max_{\iota=1,2} \frac{\mu}{r} + \frac{x-\mu}{\lambda_\iota + r} + \frac{\sigma e^{rt}}{\lambda_\iota + r} \int_t^\infty e^{-ru} \gamma_\iota du \\ &= \max_{\iota=1,2} \frac{\mu}{r} + \frac{xr - \mu r + \sigma \gamma_\iota}{(\lambda_\iota + r)\, r}. \end{aligned}$$

Remembering again the fact $\lambda_2 > \lambda_1$, we can by the same argument as above explicitly maximise over both investor groups and hence obtain the desired result. □

Similarly to the model of Chen and Kohn (see [21]), we define an *equilibrium price* as a continuous function satisfying

$$P(x,t) = \max_{\iota=1,2} \sup_{\tau \geq t} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau - t)} P(D_\tau, \tau) \kappa \middle| D_t = x \right) \tag{5.25}$$

and

$$P(x,t) \geq I(x,t) \tag{5.26}$$

where the supremum is taken over all stopping times $\tau \geq t$. Additionally, we assume $|P(D_\infty, \infty)| < \infty$. The definition above includes transaction cost compared with [21]. Since the the expectation of $\int_t^T e^{-r(s-t)} D_s ds$ is finite for every $T \geq t$, an equilibrium price always exists and is by the definition above at least the intrinsic value. Note, that an equilibrium price doesn't have to be unique. If, for instance, $P(x,t)$ is an equilibrium price, $\kappa = 1$ and $c > 0$, then $P(x,t) + ce^{rt}$ is obviously also an equilibrium price. This is the reason why we are looking for a minimal equilibrium price. Since immediate resale can obviously only be optimal in absence of transaction costs, the construction theorem for the minimal equilibrium price from [21] also holds in the Lévy case with a slight modification. Obviously, including transaction cost into the model, immediate resale can never be optimal.This changes the situation dramatically. An interesting question is also when to stop. Without resale, we can get at least the intrinsic value, but it could be more. A possible strategy the never results into a loss would be to choose the stopping time

$$\tau^* = \inf \left(T \geq t \; : \; \int_t^T e^{-r(s-t)} D_s ds + e^{-r(T-t)} P\left(D_T\right) \kappa \geq P\left(D_t\right) \right). \tag{5.27}$$

If τ^* is infinite form the viewpoint of the owner and the other group, the asset will be held forever. In other words, stopping before τ^* is equivalent to selling the asset with a loss which can never lead to an equilibrium. Thus, we can also write the definition of an equilibrium price as

$$P\left(x,t\right) = \max_{\iota=1,2} \sup_{\tau \geq \tau^*} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau-t)} P\left(D_\tau, \tau\right) \kappa \middle| D_t = x \right) \tag{5.28}$$

Now we are finally able to formulate a theorem that shows how to construct a minimal equilibrium price.

Theorem 16. *Let $I\left(x,t\right)$ be the intrinsic value. Then, we can construct a minimal equilibrium price. Therefore, define $P_0\left(x,t\right) = I\left(x,t\right)$, a sequence of stopping times*

$$\tau_k = \inf \left(T \geq t : \int_t^T e^{-r(s-t)} D_s ds + e^{-r(T-t)} P_{k-1}\left(D_T, T\right) \frac{k\kappa}{k+1} \geq P_{k-1}\left(D_t, t\right) \right) \tag{5.29}$$

and

$$P_k\left(x,t\right) = \max_{\iota=1,2} \sup_{\tau \geq \tau_k} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^{\tau_k} e^{-r(s-t)} D_s ds + e^{-r(\tau_k-t)} P_{k-1}\left(D_{\tau_k}, \tau_k\right) \frac{k\kappa}{k+1} \middle| D_t = x \right) \tag{5.30}$$

for $k \geq 1$. Then,

$$P_*\left(x\right) = P_*\left(x,t\right) = \lim_{k\to\infty} P_k\left(x,t\right) \tag{5.31}$$

is the unique minimal equilibrium price and independent of t. For $\kappa = 1$, the sequence $P_k\left(x,t\right)$ is monotonously increasing in k.

Proof. By Lemma 16 and construction, P_0 is independent of t. Analogously to Lemma 16, we can show that the conditional expectation $\mathbb{E}^{\mathbb{P}^\iota}\left(D_{t+h} | D_t = x\right)$ is also time independent for all $h > 0$. Thus, all $P_k\left(x\right) = P_k\left(x,t\right)$ and their limit are independent of the time t. Without transaction cost, it is easy to see that $\tau_k = t$ with the same idea as in [21]. However, in general, we do not even know if τ_k is finite. Let $\tilde{P}(x)$ be a minimal equilibrium price without transaction cost that is constructed as in [21] as limit of a sequence $\left(\tilde{P}_k(x)\right)_{k\geq 0}$ defined by $\tilde{P}_0 = I(x)$ and

$$\tilde{P}_k(x) = \max_{\iota=1,2} \sup_{\tau \geq t} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau-t)} \tilde{P}_{k-1}\left(D_\tau\right) \middle| D_t = x \right). \tag{5.32}$$

Due to

$$\tilde{P}_k(x) \geq \max_{\iota=1,2} \sup_{\tau \geq \tau_k} \mathbb{E}^{\mathbb{P}^\iota} \left(\tilde{P}_{k-1}(D_\tau) \middle| D_t = x \right) = \tilde{P}_{k-1}(x), \tag{5.33}$$

the constructed sequence is monotonously increasing. Using Beppo Levi's monotone convergence theorem, $\tilde{P}$ exists and is an equilibrium price. By induction, all $\tilde{P}_k$ are smaller than any equilibrium price and so is the limit. Since P_k is only monotonously increasing if $\tau_k < \infty$, we need to use another idea than monotone convergence for the general case. We show now by induction that $\tilde{P}(x) \geq P_k(x)$ for all $k \geq 1$. Obviously, $\tilde{P}(x) \geq I(x) = P_0(x)$ holds. Since $0 < \kappa \leq 1$, using $\tilde{P}(x) \geq P_{k-1}(x)$, we obtain

$$\begin{aligned}
\tilde{P}(x) &= \max_{\iota=1,2} \sup_{\tau \geq t} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau-t)} P_{k-1}(D_\tau) \middle| D_t = x \right) \\
&\geq \max_{\iota=1,2} \sup_{\tau \geq \tau_k} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau-t)} P_{k-1}(D_\tau) \frac{k\kappa}{k+1} \middle| D_t = x \right) \\
&= P_k(x).
\end{aligned}$$

Thus, $|P_k(x)| \leq \tilde{P}(x) < \infty$ for all $k \geq 1$. Since we have required that the intrinsic value to be finite, the limit

$$P_*(x) = \lim_{k \to \infty} P_k(x) \tag{5.34}$$

always exits. By the dominated convergence theorem, we get

$$P_*(x) = \max_{\iota=1,2} \sup_{\tau \geq \tau_k^*} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau-t)} P_*(D_\tau) \kappa \middle| D_t = x \right). \tag{5.35}$$

Hence, such a price P_* is always an equilibrium price. Let us now show its minimality by induction. Let $P(x,t)$ be an arbitrary equilibrium price. The initial step obviously holds by definition. Supposing $P(x,t) \geq P_{k-1}(x)$ for all $t \geq 0$, we obtain

$$\begin{aligned}
P(x,t) &= \max_{\iota=1,2} \sup_{\tau \geq \tau_k} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau-t)} P(D_\tau, \tau) \kappa \middle| D_t = x \right) \\
&\geq \max_{\iota=1,2} \sup_{\tau \geq \tau_k} \mathbb{E}^{\mathbb{P}^\iota} \left(\int_t^\tau e^{-r(s-t)} D_s ds + e^{-r(\tau-t)} P_{k-1}(D_\tau) \frac{k\kappa}{k+1} \middle| D_t = x \right) \\
&= P_k(x).
\end{aligned}$$

Taking the limit follows $P(x,t) \geq \lim_{k \to \infty} P_k(x) = P_*(x)$. Thus, $P_*(x)$ is a minimal equilibrium price. □

As in the existing literature [21, 89, 90], an *asset bubble* is defined as the difference between minimal equilibrium price and intrinsic value, i.e.

$$B(x,t) = P(x,t) - I(x,t). \tag{5.36}$$

Apparently, an asset bubble cannot be negative by definition. It is important to note that in our model the bubble is due to the α-stability of L_t^ι time-independent and will therefore never burst. In order to determine its size, we shall find another representation of the equilibrium price using the same idea as [21]. We begin with a technical lemma.

Lemma 17. *The Lévy measure of D_t satisfies $\nu_D(A) = \sigma^\alpha \nu(A)$.*

Proof. First, we compute the jumps

$$\Delta D_t = D_t - D_{t-} = \sigma \int_{t-}^{t} e^{-\lambda_\iota(t-s)} dL_s^\iota = \sigma\left(L_t^\iota - L_{t-}^\iota\right) = \sigma \Delta L_t^\iota. \tag{5.37}$$

Hence, we get

$$\begin{aligned}\nu_D(A) &= \mathbb{E}^{\mathbb{P}^\iota}\left(\#\{t \in [0,1] : \Delta D_t \neq 0, \Delta D_t \in A\}\right)\\ &= \mathbb{E}^{\mathbb{P}^\iota}\left(\#\left\{t \in [0,1] : \Delta L_t^\iota \neq 0, \Delta L_t^\iota \in \sigma^{-1} A\right\}\right) = \nu(\sigma^{-1}A).\end{aligned}$$

We remember the well known representation of the Lévy measure for α-stable processes

$$\nu(dx) = \frac{c_1 \mathbf{1}_{x<0}}{|x|^{\alpha+1}} dx + \frac{c_2 \mathbf{1}_{x>0}}{x^{\alpha+1}} dx \tag{5.38}$$

with $c_1 \geq 0$, $c_2 \geq 0$ and $c_1 + c_2 > 0$. By a linear substitution, we obtain

$$\begin{aligned}\nu(\sigma^{-1}dx) = \nu(dy) &= \frac{c_1 \mathbf{1}_{y<0}}{|y|^{\alpha+1}} dy + \frac{c_2 \mathbf{1}_{y>0}}{y^{\alpha+1}} dy\\ &= \frac{c_1 \mathbf{1}_{x<0}}{|\frac{x}{\sigma}|^{\alpha+1}} \frac{dx}{\sigma} + \frac{c_2 \mathbf{1}_{x>0}}{(\frac{x}{\sigma})^{\alpha+1}} \frac{dx}{\sigma} = \sigma^\alpha \nu(dx).\end{aligned}$$

□

According to Lemma 17 and Lemma 4, the process $(\sigma L_t^\iota)_{t\geq 0}$ is again a Lévy process with an α-stable jumps having the characteristic triplet $(\breve{a}, \breve{\nu}, \breve{\gamma}_\iota)$ where

$$\begin{aligned}\breve{a} &= \sigma a,\\ \breve{\nu}(dx) &= \sigma^\alpha \nu(dx) = \nu_D(dx),\\ \breve{\gamma}_\iota &= \begin{cases} \sigma\left(\gamma_\iota - \frac{\sigma^{\alpha-1}-1}{\alpha-1}(c_2 - c_1)\right) & \text{if } \alpha \neq 1,\\ \sigma\left(\gamma_\iota - \log(\sigma)(c_2 - c_1)\right) & \text{if } \alpha = 1\end{cases}.\end{aligned}$$

By Lemma 17 we get the important relation $\breve{\nu}(dx) = \nu_D(dx)$. Collecting now all our information, we can find a pseudo-differential equation such that its solution, if it exists, is an equilibrium price. As for calculating the intrinsic value, also in this proof α-stability is a crucial assumption which allows us to characterise the jumps part of the dividend process and hence construct a link to pseudo-differential equations via the Itô formula.

Theorem 17. *Let $\Phi(x,t) : \mathbb{R} \times [0, \infty,) \to \mathbb{R}$ be a twice continuously differentiable solution of the pseudo-differential equation*

$$\begin{aligned}\max_{\iota=1,2}\Bigg(\left(\lambda_\iota(\mu - x) - \sigma(\gamma_\iota + \frac{c_2 - c_1}{\alpha - 1}\mathbf{1}_{\alpha\in(1,2)})\right)\frac{\partial}{\partial x}\Phi(x,t) +&\\ \frac{\partial}{\partial t}\Phi(x,t) - r\Phi(x,t) + \mathcal{A}_\iota\Phi(x,t) + \frac{x}{\kappa}\Bigg) &= 0\end{aligned} \tag{5.39}$$

where c_1 and c_2 are determined by the Lévy-measure of L_t^ι and the infinitesimal generator of $(\sigma L_t^\iota)_{t\geq 0}$ can be written as

$$\begin{aligned}\mathcal{A}_\iota\Phi(x,t) = \breve{\gamma}_\iota \Phi_x(x,t) &+ \frac{\breve{a}^2}{2}\Phi_{xx}(x,t)\\ &+ \int_{\mathbb{R}\setminus\{0\}} \left(\Phi(x+z,t) - \Phi(x,t) - \Phi_x(x,t) z \mathbf{1}_{|z|\leq 1}\right) \breve{\nu}(dz).\end{aligned}$$

In the case $\alpha \geq 1$, we additionally assume that L_t^ι is symmetric. Then, $\Phi(x,t)$ is an upper bound for an equilibrium price and the strategy

$$\tau^* = \inf\left(T \geq t : \int_t^T e^{-r(s-t)} D_s ds + e^{-r(T-t)}\Phi\left(D_T, T\right)\kappa \geq \Phi\left(D_t, t\right)\right) \tag{5.40}$$

is optimal. If $\tau^* < \infty$, then $\Phi(x,t)$ is an equilibrium price.

Proof. As a first step, we analyse the large jumps. The case $\alpha = 2$ is trivial since there are no jumps. For $0 < \alpha < 1$, we can interpret the large jumps as part of the drift. Using Lemma 17, we can compute

$$\int_{\mathbb{R}\setminus\{0\}} z\mathbf{1}_{|z|>1}\nu_D(dz) = \int_{-\infty}^{-1} \frac{c_1\sigma^\alpha}{(-z)^\alpha}dz + \int_1^\infty \frac{c_2\sigma^\alpha}{z^\alpha}dz = \frac{c_2 - c_1}{\alpha - 1}\sigma^\alpha. \tag{5.41}$$

In the case $\alpha = 1$ we make use of the symmetry and obtain

$$\int_{\mathbb{R}\setminus\{0\}} z\mathbf{1}_{|z|>1}\nu_D(dz) = \int_{-\infty}^{-1} \frac{c\sigma^\alpha}{z}dz + \int_1^\infty \frac{c\sigma^\alpha}{z}dz = 0. \tag{5.42}$$

By the same reason, the case $\alpha > 1$ leads to

$$\int_{\mathbb{R}\setminus\{0\}} z\mathbf{1}_{|z|>1}\nu_D(dz) = \int_{-\infty}^{-1} \frac{-c\sigma^\alpha}{(-z)^\alpha}dz + \int_1^\infty \frac{c\sigma^\alpha}{z^\alpha}dz = \frac{c - c}{\alpha - 1}\sigma^\alpha = 0. \tag{5.43}$$

The second step shows the link to pseudo-differential equations. Applying the Itô formula (see [83, p. 78f]) onto $f(D_t,t) := e^{-rt}\Phi(D_t,t)$ and plugging in the stochastic differential equation, we obtain

$$\begin{aligned}
e^{-rT}\Phi(D_T,T) &= e^{-rt}\Phi(D_t,t) \\
&\quad + \int_t^T e^{-rs}\Phi_x(D_{s-},s)dD_s + \int_t^T e^{-rs}\left(\Phi_t(D_{s-},s) - r\Phi(D_{s-},s)\right)ds \\
&\quad + \frac{1}{2}\int_t^T e^{-rs}\Phi_{xx}(D_{s-},s)d[D,D]_s^c \\
&\quad + \sum_{t\leq s\leq T}\left(e^{-rs}\Phi(D_s,s) - e^{-rs}\Phi(D_{s-},s) - e^{-rs}\Phi_x(D_{s-},s)\Delta D_s\right) \\
&= e^{-rt}\Phi(D_t,t) + \int_t^T e^{-rs}\Phi_x(D_{s-},s)\lambda_\iota\left(\mu - D_{s-}\right)ds \\
&\quad + \int_t^T e^{-rs}\Phi_x(D_{s-},s)\sigma dL_s^\iota + \int_t^T e^{-rs}\left(\Phi_t(D_{s-},s) - r\Phi(D_{s-},s)\right)ds \\
&\quad + \frac{1}{2}\int_t^T e^{-rs}\Phi_{xx}(D_{s-},s)d[D,D]_s^c \\
&\quad + \int_t^T\int_{\mathbb{R}\setminus\{0\}} e^{-rs}\left(\Phi(D_{s-}+z,s) - \Phi(D_{s-},s) - \Phi_x(D_{s-},s)z\right)J_D(ds,dz)
\end{aligned}$$

$$
\begin{aligned}
&= e^{-rt}\Phi(D_t,t) \\
&+ \int_t^T e^{-rs}\left(\lambda_\iota\left(\mu - D_{s-}\right)\Phi_x(D_{s-},s) + \Phi_t(D_{s-},s) - r\Phi(D_{s-},s)\right)ds \\
&+ \int_t^T e^{-rs}\Phi_x(D_{s-},s)\sigma dL_s^\iota + \frac{1}{2}\sigma^2\int_t^T e^{-rs}\Phi_{xx}(D_{s-},s)d[L^\iota,L^\iota]_s^c \\
&+ \int_t^T\int_{\mathbb{R}\setminus\{0\}} e^{-rs}\left(\Phi(D_{s-}+z,s) - \Phi(D_{s-},s) - \Phi_x(D_{s-},s)z\right)\tilde{J}_D(ds,dz) \\
&+ \int_t^T\int_{\mathbb{R}\setminus\{0\}} e^{-rs}\left(\Phi(D_{s-}+z,s) - \Phi(D_{s-},s) - \Phi_x(D_{s-},s)z\mathbf{1}_{|z|\le 1}\right)\nu_D(dz)ds \\
&- \int_t^T e^{-rs}\Phi_x(D_{s-},s)\int_{\mathbb{R}\setminus\{0\}} z\mathbf{1}_{|z|>1}\nu_D(dz)ds.
\end{aligned}
$$

Now we use the properties of the quadratic variation, rewrite the large jumps as shown in the first step of this proof and after rearranging the terms, we receive

$$
\begin{aligned}
e^{-rT}\Phi(D_T,T) &= e^{-rt}\Phi(D_t,t) + \int_t^T e^{-rs}\Phi_x(D_{s-},s)\sigma dL_s^\iota \\
&+ \int_t^T e^{-rs}\left(\left(\lambda_\iota\left(\mu - D_{s-}\right) - \sigma^\alpha\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(0,1)}\right)\Phi_x(D_{s-},s)\right)ds \\
&+ \int_t^T e^{-rs}\left(\Phi_t(D_{s-},s) - r\Phi(D_{s-},s) + \frac{\sigma^2 a^2}{2}\Phi_{xx}(D_{s-},s)\right)ds \\
&+ \int_t^T e^{-rs}\int_{\mathbb{R}\setminus\{0\}}\left(\Phi(D_{s-}+z,s) - \Phi(D_{s-},s) - \Phi_x(D_{s-},s)z\mathbf{1}_{|z|\le 1}\right)\check{\nu}(dz)ds \\
&+ \int_t^T\int_{\mathbb{R}\setminus\{0\}} e^{-rs}\left(\Phi(D_{s-}+z,s) - \Phi(D_{s-},s) - \Phi_x(D_{s-},s)z\right)\tilde{J}_D(ds,dz).
\end{aligned}
$$

In the next step, let us assume that equation (5.39) holds for $\Phi(x,t)$. The terms integrated with respect to ds simplify and so we get

$$
\begin{aligned}
e^{-rT}\Phi(D_T,T) &\le e^{-rt}\Phi(D_t,t) - \int_t^T e^{-rs}\frac{D_{s-}}{\kappa}ds + \int_t^T e^{-rs}\Phi_x(D_{s-},s)\sigma dL_s^\iota \\
&+ \int_t^T\int_{\mathbb{R}\setminus\{0\}} e^{-rs}\left(\Phi(D_{s-}+z,s) - \Phi(D_{s-},s) - \Phi_x(D_{s-},s)z\right)\tilde{J}_D(ds,dz).
\end{aligned}
$$

Since the integral with respect to the compensated Poisson random measure $\tilde{J}_D$ and the integral with respect to the Lévy process L^ι vanish by taking the expectation, we get the inequality

$$
\mathbb{E}^{\mathbb{P}^\iota}\left(e^{-rT}\Phi(D_T,T)\middle|D_t = x\right) \le e^{-rt}\Phi(x,t) - \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^T e^{-ru}\frac{D_{u-}}{\kappa}du\middle|D_t = x\right). \tag{5.44}
$$

Rearranging leads to

$$
\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^T e^{-r(u-t)}D_{u-}du + e^{-r(T-t)}\Phi(D_T,T)\kappa\middle|D_t = x\right) \le \Phi(x,t) \tag{5.45}
$$

for all $T \ge t$. Thus, the inequality also holds taking the supremum over all stopping times and after the maximising we get

$$
\Phi(x,t) \ge \max_{\iota=1,2}\sup_{\tau\ge t}\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^\tau e^{-r(u-t)}D_{u-}du + e^{-r(\tau-t)}\Phi(D_\tau,\tau)\kappa\middle|D_t = x\right). \tag{5.46}
$$

Choosing the stopping time τ^* leads to

$$\begin{aligned}\Phi(x,t) &\geq \max_{\iota=1,2} \mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^{\tau^*} e^{-r(u-t)} D_{u-} du + e^{-r(\tau^*-t)}\Phi(D^*_\tau,\tau^*)\kappa \middle| D_t = x\right)\\ &\geq \Phi(x,t)\end{aligned}$$

if $\tau^* < \infty$. From inequality (5.45) we obtain $\Phi(x,t) \geq I(x,t)$ setting $T \to \infty$. Hence, $\Phi(x,t)$ is an equilibrium price for $\tau^* < \infty$ and in general obviously an upper bound for an equilibrium price. □

Obviously, when $\kappa = 1$, the optimal strategy is immediate stopping at $\tau = t$. In the case when L^ι is a Brownian motion, the infinitesimal generator is $\frac{\sigma^2 a^2}{2}\Delta_x$, where Δ is the Laplacian (see [88, p. 212]). Then, the equation turns into a partial differential equation and assuming time independence into a Weber differential equation (see the model by Chen and Kohn [21]). However, in the more general case, the pseudo-differential equation from Theorem 17 can be written as

$$\frac{\partial}{\partial t}\Phi(x,t) = -T_x\Phi(x,t) - \frac{x}{\kappa} \tag{5.47}$$

where T_x is a pseudo-differential operator (see Section 2.4) with symbol

$$\begin{aligned}p(x;\xi) = -r + &\left(\max_{j=1,2} \lambda_j(\mu - x) + \frac{\sigma^\alpha(c_2-c_1)}{\alpha-1}\mathbf{1}_{\alpha\in(0,1)}\right) i\xi\\ &+ (c_1+c_2)\sigma^\alpha|\xi|^\alpha\left(1 - i\frac{c_1-c_2}{c_1+c_2}\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sgn}(\xi)\right).\end{aligned} \tag{5.48}$$

We remark that in the case where both processes L^1 and L^2 are symmetric, the generator simplifies to

$$\mathcal{A}_\iota = -(-\Delta)^{\frac{\alpha}{2}} \tag{5.49}$$

where Δ denotes the one-dimensional Laplacian. In the general case, taking a close look at the form of the equation, shows us that we are in fact dealing with a partial integro differential equation (PIDE). The is quite common in Lévy modelling and can be found in option pricing theory (see Chapter 12 in [24]). Solving it is of course more complicated than solving a partial differential equation. Due to Theorem 16, we know that a minimal equilibrium price is time independent. Hence, we additionally suppose

$$\frac{\partial}{\partial t}\Phi(x,t) = 0 \tag{5.50}$$

and we further write $\Phi(x) = \Phi(x,t)$. Therefore, we obtain the integro differential equation

$$\begin{aligned}\max_{\iota=1,2}&\left(\lambda_\iota(\mu - x) - \sigma^\alpha\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(1,2)}\right)\Phi'(x) - r\Phi(x) + \frac{\breve{a}^2}{2}\Phi''(x)\\ &+ \sigma^\alpha\int_{\mathbb{R}\setminus\{0\}}\left(\Phi(x+z) - \Phi(x) - \Phi'(x)z\mathbf{1}_{|z|\leq 1}\right)\nu(dz) + \frac{x}{\kappa} = 0.\end{aligned} \tag{5.51}$$

As the coefficients are not constant, following a Fourier transformation approach doesn't simplify the problem. So, we need another idea to examine the existence of a solution. In a next step, we set

$$\Psi(x,t) = \Phi(x) - \frac{x}{\kappa(\lambda_i + r)} - \frac{\mu\lambda_i}{r\kappa(\lambda_i+r)} \tag{5.52}$$

and our equation turns into

$$\max_{\iota=1,2}\left(\lambda_\iota\left(\mu-x\right)-\sigma^\alpha\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(1,2)}\right)\Psi'(x)-r\Psi(x)+\frac{\breve{a}^2}{2}\Psi''(x)$$
$$+\sigma^\alpha\int_{\mathbb{R}\setminus\{0\}}\left(\Psi(x+z)-\Psi(x)-\Psi'(x)z\mathbf{1}_{|z|\le 1}\right)\nu(dz)=0. \quad (5.53)$$

We write this as pseudo differential equation

$$(T_\Psi-r)\Psi(x,t)=0 \quad (5.54)$$

where T_Ψ is a pseudo-differential operator with symbol

$$p\left(x;\xi\right)=\left(\max_{j=1,2}\lambda_j(\mu-x)+\frac{\sigma^\alpha(c_2-c_1)}{\alpha-1}\mathbf{1}_{\alpha\in(0,1)}\right)i\xi$$
$$+(c_1+c_2)\sigma^\alpha\left|\xi\right|^\alpha\left(1-i\frac{c_1-c_2}{c_1+c_2}\tan\left(\frac{\pi\alpha}{2}\right)\mathrm{sgn}\left(\xi\right)\right). \quad (5.55)$$

Due to

$$\mathrm{Re}\left(p\left(x;\xi\right)\right)=(c_1+c_2)\sigma^\alpha\left|\xi\right|^\alpha \quad (5.56)$$

the operator is strictly elliptic. Since $r>0$ and the function constantly equal to 0 is in $L^2\left(\mathbb{R}\right)$, we can apply Theorem 8 and receive the existence of a unique strong solution in $L^2\left(\mathbb{R}\right)$. However, we do not know the asymptotic behaviour at infinity.

5.3. The Equilibrium Price as a Solution of a PIDE

Now let us suppose that there is a solution with linear behaviour at infinity. We will show that $\Phi(y)$ from Theorem 17 is a minimal equilibrium price. Since the theory of viscosity solutions can be generalised to pseudo differential equations (see chapter 2.5), we can carefully adapt [21] and [22] to a Lévy setting. The main idea from stochastic control theory is a verification argument: we take a solution of a differential equation with certain properties and show that this is also the solution of an optimisation problem. We start with proving that P_* defined in Theorem 16 is a viscosity supersolution.

Lemma 18. *P_* is lower semicontinuous.*

Proof. Let $(x_k)_{k\ge 0}$ be a sequence converging to x. Let us define a sequence of stopping times

$$\tilde{\tau}_k=\inf\left\{s\ge t\ :\ D_s=x,D_t=x_k\right\}. \quad (5.57)$$

that obviously converges to t. Since $P_*\left(x\right)$ is an equilibrium price, for $\iota=1,2$ holds

$$\liminf_{k\to\infty}P_*\left(x_k\right)\ge$$
$$\liminf_{k\to\infty}\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^{\tilde{\tau}_k}e^{-r(s-t)}D_s ds\middle|D_t=x_k\right)+\liminf_{k\to\infty}\mathbb{E}^{\mathbb{P}^\iota}\left(e^{-r(\tilde{\tau}_k-t)}P_*\left(x\right)\middle|D_t=x_k\right) \quad (5.58)$$

By the dominated convergence theorem, the first expectation is zero and hence, we get

$$\liminf_{k\to\infty}P_*\left(x_k\right)\ge P_*\left(x\right). \quad (5.59)$$

□

Using the lower semicontinuity, we are now able to show the next lemma.

Lemma 19. *P_* is a viscosity supersolution.*

Proof. We will prove it by contradiction. Let us suppose P_* is a not viscosity supersolution. Then there exists $\psi(x) \in C^2(\mathbb{R}) \cap \mathcal{C}$ and local maximum point $\hat{x}$ of $\psi(x) - P_*(x)$ that satisfies $\psi(\hat{x}) = P_*(\hat{x})$ and

$$-\max_{\iota=1,2}\left(\left(\lambda_\iota\left(\mu-\hat{x}\right)-\sigma(\gamma_\iota+\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(1,2)})\right)\psi'(\hat{x}) \right.$$
$$\left. -rP_*(\hat{x})+\mathcal{A}_\iota\psi(\hat{x})+\frac{\hat{x}}{\kappa}\right) \leq -\delta. \quad (5.60)$$

for a $\delta > 0$. For $\epsilon > 0$ let us choose an interval $[\hat{x}-\epsilon, \hat{x}+\epsilon]$ on which $\psi(x) - P_*\left(x\right) \leq 0$ and

$$\max_{\iota=1,2}\left(\left(\lambda_\iota\left(\mu-x\right)-\sigma(\gamma_\iota+\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(1,2)})\right)\psi'(x) \right.$$
$$\left. -rP_*(x)+\mathcal{A}_\iota\psi(x)+\frac{x}{\kappa}\right) \geq \delta. \quad (5.61)$$

We define a stopping time

$$\tilde{\tau} = \inf\left\{s \geq t \;:\; D_s = \hat{x}-\epsilon \;\vee\; D_s = \hat{x}+\epsilon, D_t = \hat{x}\right\}. \quad (5.62)$$

as first time hitting the interval border. Obviously, $P(\tilde{\tau} > 0) = 1$. With the same idea as in Theorem 17, we apply the Itô formula onto

$$e^{-r(s-t)}\psi(D_s). \quad (5.63)$$

After integrating and ignoring the terms integrated with respect to martingales that are zero, we get

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\iota}\left(e^{-r(\tilde{\tau}-t)}\psi(D_{\tilde{\tau}})\Big|D_t=\hat{x}\right)\\
&\quad=\psi(\hat{x})+\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^{\tilde{\tau}}e^{-r(s-t)}\left(\left(\lambda_\iota\left(\mu-x\right)-\sigma(\gamma_\iota+\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(1,2)})\right)\psi'(x)\right.\right.\\
&\qquad\left.\left.-r\psi(x)+\mathcal{A}_\iota\psi(x)+\frac{x}{\kappa}\right)ds\Big|D_t=\hat{x}\right)
\end{aligned}$$

for both $\iota \in \{1,2\}$. Since $r > 0$ and due to lower semicontinuity $\psi(x) \leq P_*\left(x\right)$, we receive

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\iota}\left(e^{-r(\tilde{\tau}-t)}\psi(D_{\tilde{\tau}})\Big|D_t=\hat{x}\right)\\
&\quad=\psi(\hat{x})+\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^{\tilde{\tau}}e^{-r(s-t)}\left(\left(\lambda_\iota\left(\mu-x\right)-\sigma(\gamma_\iota+\frac{c_2-c_1}{\alpha-1}\mathbf{1}_{\alpha\in(1,2)})\right)\psi'(x)\right.\right.\\
&\qquad\left.\left.-rP_*(x)+\mathcal{A}_\iota\psi(x)+\frac{x}{\kappa}\right)ds\Big|D_t=\hat{x}\right)\\
&\quad\geq\psi(\hat{x})+\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^{\tilde{\tau}}e^{-r(s-t)}\delta ds\Big|D_t=\hat{x}\right)>\psi(\hat{x}).
\end{aligned}$$

On the other hand, $P_*(x)$ is an equilibrium price and hence

$$P_*\left(\hat{x}\right) \geq \max_{\iota=1,2}\mathbb{E}^{\mathbb{P}^\iota}\left(\int_t^{\tau}e^{-r(s-t)}D_s ds+e^{-r(\tau-t)}P_*\left(D_\tau\right)\kappa\Big|D_t=\hat{x}\right) \quad (5.64)$$

Putting everything together with $P_*(x) - \psi(x) \geq 0$, we obtain

$$0 = P_*(\hat{x}) - \psi(\hat{x}) > \max_{\iota \in \{1,2\}} \mathbb{E}^{\mathbb{P}^\iota} \left(e^{-r(\tilde{\tau}-t)} \left(P_*(D_{\tilde{\tau}}) - \psi(D_{\tilde{\tau}}) \right) \middle| D_t = \hat{x} \right) \geq 0, \tag{5.65}$$

which is a contradiction. Hence $P_*(x)$ is a viscosity supersolution, as it is also lower semicontinuous by Lemma 18. □

The next theorem shows the equivalence between the minimal equilibrium price and the solution of a partial integro-differential equation. The crucial step in the proof uses at most linear growth of an equilibrium price at infinity.

Theorem 18. *The unique solution of the partial integro-differential equation with linear growth at infinity is the minimal equilibrium price, in other words $\Phi(x) = P_*(x)$.*

Proof. Obviously $P_*(x) \leq \Phi(x)$ holds, because $P_*(x)$ is the minimal equilibrium price and $\Phi(x)$ is an equilibrium price. Analogously to [22], we examine

$$\inf_{x \in \mathbb{R}} \left(P_*(x) - \Phi(x) \right) \tag{5.66}$$

and consider two cases. First, let the infimum be unbounded. Since we assumed linear growth of $\Phi(x)$ and we know $\Phi(x) - P_*(x) \leq \Phi(x) - I(x)$, we obtain

$$\lim_{|x| \to \infty} \left(P_*(x) - \Phi(x) \right) = 0. \tag{5.67}$$

In the bounded case, there exists a minimal point $\hat{x} \in \mathbb{R}$. Due to Lemma 19, $P_*(y)$ is a viscosity supersolution and hence,

$$-\max_{\iota=1,2} \left(\lambda_\iota (\mu - \hat{x}) - \sigma(\gamma_\iota + \frac{c_2 - c_1}{\alpha - 1} \mathbf{1}_{\alpha \in (1,2)}) \right) \Phi'(\hat{x}) - rP_*(\hat{x}) + \mathcal{A}\Phi(\hat{x}) + \frac{\hat{x}}{\kappa} \geq 0. \tag{5.68}$$

As $\Phi(x)$ is a strong solution of a partial integro differenial equation and $P_*(x) - \Phi(x) \leq 0$, we get

$$-\max_{\iota \in \{1,2\}} \left(rP_*(\hat{x}) - r\Phi(\hat{x}) \right) \geq 0 \tag{5.69}$$

As $r > 0$ and $\hat{x}$ is the minimal point of $P_*(x) - \Phi(x)$, we obtain

$$P_*(x) - \Phi(x) \geq P_*(\hat{x}) - \Phi(\hat{x}) \geq 0 \tag{5.70}$$

and finally $P_*(x) = \Phi(x)$. □

The interesting point is to know if there in a bubble in our market. We remember the definition of a bubble as

$$B(x) = \Phi(x) - I(x).$$

There is unfortunately very few literature about how to handle PIDE numerically. We could discretisise like [85], but in our case the coefficients are not constant. Therefore, the situation is much more complicated and it can be an interesting field for future research. Since we include transaction costs, the investors have a different optimal strategy than in a simple model. Immediate resale is no longer optimal. They wait until the resale value reaches at least the price they have paid plus the transaction fee. This also has an effect on the bubble. For large transaction costs, the investors trade less and, as a consequence, the bubble becomes smaller.

A. Some Special Matrix Functions

This chapter introduces some special functions with matrix parameters. Most of their properties and the link to matrix differential equations were discussed in several papers by Jódar and Cortés [65–67]. The main difficulty in introducing such functions is the invertibility of the involed matrices. Let $\mathtt{L}$ and $\mathtt{M}$ be $N \times N$ matrices. First, we define the **Pochhammer symbol** for matrices as

$$(\mathtt{M})_k = (\mathtt{M} + (k-1)\mathtt{I}) \dots (\mathtt{M} + \mathtt{I})\,\mathtt{M} \quad \text{for } k \geq 1,$$
$$(\mathtt{M})_0 = \mathtt{I}.$$

We call a matrix positive stable, if it has only eigenvalues with positive real part. The matrix exponential allows us to define

$$t^{\mathtt{M}} = e^{\mathtt{M} \ln t} = \sum_{k=0}^{\infty} \mathtt{M}^k \frac{(\ln t)^k}{k!}.$$

The **Gamma matrix function** for positive stable matrices $\mathtt{M}$ has been introduced by [66] as

$$\Gamma(\mathtt{M}) = \int_0^\infty e^{-t} t^{\mathtt{M}-\mathtt{I}} dt.$$

By the help of infinite matrix products, the Gamma matrix function has been extended (see [27]) to matrices with only non-negative-integer eigenvalues, i.e. $-n \notin \sigma(\mathtt{M})$ for $n \in \mathbb{N} \setminus \{0\}$. If $\mathtt{M} + n\mathtt{I}$ is an invertible matrix for every integer $n \geq 0$, then $\Gamma(\mathtt{M})$ is also invertible and its inverse corresponds to the inverse of the Gamma function [66]. The **Beta matrix function** for positive stable matrices $\mathtt{L}$ and $\mathtt{M}$ is defined as

$$B(\mathtt{L}, \mathtt{M}) = \int_0^1 t^{\mathtt{L}-\mathtt{I}} (1-t)^{\mathtt{M}-\mathtt{I}} dt.$$

It can be shown that $B(\mathtt{L}, \mathtt{M})$ is symmetric if and only if $\mathtt{L}$ and $\mathtt{M}$ are commuting matrices [66]. The following lemma describes the relationship between Beta and Gamma matrix function.

Lemma 20. *For positive stable, commuting matrices* $\mathtt{L}$ *and* $\mathtt{M}$ *such that* $\mathtt{L} + \mathtt{M}$ *has only non-negative-integer eigenvalues holds*

$$B(\mathtt{L}, \mathtt{M}) = \Gamma(\mathtt{L})\Gamma(\mathtt{M})\Gamma(\mathtt{L} + \mathtt{M})^{-1}.$$

Proof. First, we write

$$\begin{aligned} \Gamma(\mathtt{L})\Gamma(\mathtt{M}) &= \left(\int_0^\infty e^{-s} s^{\mathtt{L}-\mathtt{I}} ds \right) \left(\int_0^\infty e^{-t} t^{\mathtt{M}-\mathtt{I}} dt \right) \\ &= \int_0^\infty \int_0^\infty e^{-s} s^{\mathtt{L}-\mathtt{I}} e^{-t} t^{\mathtt{M}-\mathtt{I}} ds dt. \end{aligned}$$

The change of variables $x = \frac{s}{s+t}$ and $y = s + t$ and commutativity lead to

$$\begin{aligned}
\Gamma(\mathtt{L})\Gamma(\mathtt{M}) &= \int_0^\infty \int_0^1 e^{-xy}(xy)^{\mathtt{L}-\mathtt{I}} e^{-y(1-x)}(y(1-x))^{\mathtt{M}-\mathtt{I}} y dx dy. \\
&= \left(\int_0^\infty e^{-y} y^{\mathtt{L}+\mathtt{M}-\mathtt{I}} dy\right)\left(\int_0^1 x^{\mathtt{L}-\mathtt{I}}(x-1)^{\mathtt{M}-\mathtt{I}} dy\right) \\
&= \Gamma(\mathtt{L}+\mathtt{M})B(\mathtt{L},\mathtt{M}).
\end{aligned}$$

Note that the condition that $\mathtt{L}+\mathtt{M}$ has to be positive stable from Lemma 2 in [65] is not necessary. The extension of th Gamma function by [27] makes $\Gamma(\mathtt{L}+\mathtt{M})$ is well-defined and, hence, also invertible. □

We repeat the integral representation of the Pochhammer matrix symbol from [65] in the next lemma.

Lemma 21. *For positive stable, commuting matrices* $\mathtt{L}$ *and* $\mathtt{M}$*, such that* $\mathtt{M}-\mathtt{L}$ *is also positive stable, holds*

$$(\mathtt{L})_k(\mathtt{M})_k^{-1} = \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})^{-1}\left(\int_0^1 t^{\mathtt{L}+(k-1)\mathtt{I}}(1-t)^{\mathtt{M}-\mathtt{L}-\mathtt{I}} dt\right)\Gamma(\mathtt{M})$$

for every $k \in \mathbb{N}$.

Proof. From Lemma 20, we get

$$\begin{aligned}
(\mathtt{L})_k(\mathtt{M})_k^{-1} &= \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{L}+k\mathtt{I})\Gamma(\mathtt{M})\Gamma(\mathtt{M}+k\mathtt{I})^{-1} \\
&= \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{L}+k\mathtt{I})\Gamma(\mathtt{M}+k\mathtt{I})^{-1}\Gamma(\mathtt{M}) \\
&= \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})\Gamma(\mathtt{L}+k\mathtt{I})\Gamma(\mathtt{M}+k\mathtt{I})^{-1}\Gamma(\mathtt{M}) \\
&= \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})^{-1}B(\mathtt{L}+k\mathtt{I},\mathtt{M}-\mathtt{L})\Gamma(\mathtt{M}) \\
&= \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})^{-1}\left(\int_0^1 t^{\mathtt{L}+(k-1)\mathtt{I}}(1-t)^{\mathtt{M}-\mathtt{L}-\mathtt{I}} dt\right)\Gamma(\mathtt{M}).
\end{aligned}$$

□

The confluent hypergeometric function with matrix parameters, also called first Kummer function, is defined as

$${}_1F_1(\mathtt{L};\mathtt{M};z) = \sum_{k=0}^\infty (\mathtt{L})_k (\mathtt{M})_k^{-1} \frac{z^k}{k!}.$$

for $z \in \mathbb{C}$ (see [10]). From now, let $\mathtt{ML} = \mathtt{LM}$. If $\mathtt{M}-\mathtt{L}$ is positive stable, using Lemma 21, we receive

$$\begin{aligned}
{}_1F_1(\mathtt{L};\mathtt{M};z) &= \sum_{k=0}^\infty \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})^{-1}\left(\int_0^1 t^{\mathtt{L}+(k-1)\mathtt{I}}(1-t)^{\mathtt{M}-\mathtt{L}-\mathtt{I}} dt\right)\Gamma(\mathtt{M})\frac{z^k}{k!} \\
&= \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})^{-1}\left(\int_0^1 t^{\mathtt{L}-\mathtt{I}}\sum_{k=0}^\infty \frac{(tz)^k}{k!}(1-t)^{\mathtt{M}-\mathtt{L}-\mathtt{I}} dt\right)\Gamma(\mathtt{M}).
\end{aligned}$$

This gives us the integral representation

$${}_1F_1(\mathtt{L};\mathtt{M};z) = \Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{M}-\mathtt{L})^{-1}\left(\int_0^1 e^{zt} t^{\mathtt{L}-\mathtt{I}}(1-t)^{\mathtt{M}-\mathtt{L}-\mathtt{I}} dt\right)\Gamma(\mathtt{M}).$$

for all $z \in \mathbb{C}$. Since $\mathtt{L}$ commutes with $(\mathtt{M} + k\mathtt{I})^{-1}$ for all integers $k \geq 0$, we can compute

$$\begin{aligned}\frac{d}{dz}{}_1F_1(\mathtt{L};\mathtt{M};z) &= \sum_{k=1}^{\infty} (\mathtt{L})_k\,(\mathtt{M})_k^{-1}\,\frac{z^{k-1}}{(k-1)!}\\ &= \sum_{k=0}^{\infty} (\mathtt{L}+\mathtt{I})_k\,\mathtt{L}\,(\mathtt{M}+\mathtt{I})_k^{-1}\,\mathtt{M}^{-1}\frac{z^k}{k!}\\ &= {}_1F_1(\mathtt{L}+\mathtt{I};\mathtt{M}+\mathtt{I};z)\,\mathtt{L}\mathtt{M}^{-1}.\end{aligned}$$

For $k \geq 0$ we obtain

$$\frac{d^k}{dz^k}{}_1F_1(\mathtt{L};\mathtt{M};z) = {}_1F_1(\mathtt{L}+k\mathtt{I};\mathtt{M}+k\mathtt{I};z)\,(\mathtt{L})_k\,(\mathtt{M})_k^{-1}\,.$$

Let $b \in \mathbb{R} \setminus \mathbb{Z}^-$. We define the **second Kummer function** with matrix parameters as

$$\begin{aligned}\mathrm{U}(\mathtt{L}, b\mathtt{I}, z) = \Gamma\,(1-b)\,{}_1F_1(\mathtt{L}; b\mathtt{I}; z)\,\Gamma\,(\mathtt{L} + (1-b)\mathtt{I})^{-1}&\\ + z^{1-b}\Gamma(b-1){}_1F_1((1-b)\,\mathtt{I} + \mathtt{L}; (2-b)\mathtt{I}; z)\,\Gamma\,(\mathtt{L})^{-1}&.\end{aligned}$$

for $z \in \mathbb{C}$. We further introduce the matrix function

$$\begin{aligned}\mathrm{F}_{\mathtt{Y}}(z) = \Gamma\left(\frac{1}{2}\right){}_1F_1\left(-\frac{1}{2}\mathtt{Y}; \frac{1}{2}\mathtt{I}; \frac{z^2}{2}\right)\Gamma\left(\frac{1}{2}\,(\mathtt{I}-\mathtt{Y})\right)^{-1}&\\ + \frac{|z|\Gamma(-\frac{1}{2})}{\sqrt{2}}{}_1F_1\left(\frac{1}{2}\,(\mathtt{I}-\mathtt{Y})\,; \frac{3}{2}\mathtt{I}; \frac{z^2}{2}\right)\Gamma\left(-\frac{1}{2}\mathtt{Y}\right)^{-1}&,\end{aligned}$$

for $z \in \mathbb{C}$. As we see that $\mathrm{F}_{\mathtt{Y}}(z) = \mathrm{U}\left(-\frac{1}{2}\mathtt{Y}, \frac{1}{2}\mathtt{I}, \frac{z^2}{2}\right)$ we call

$$\mathrm{D}_{\mathtt{Y}}(z) = 2^{\frac{\mathtt{Y}}{2}} e^{-\frac{z^2}{4}}\,\mathrm{F}_{\mathtt{Y}}(z)$$

the **parabolic cylinder function** with matrix parameters (compare to [19, p. 39]). We compute

$$\begin{aligned}\frac{d}{dz}\,\mathrm{F}_{\mathtt{Y}}(z) = \Gamma\left(\frac{1}{2}\right){}_1F_1\left(\frac{1}{2}\,(2\mathtt{I}-\mathtt{Y})\,; \frac{3}{2}\mathtt{I}; \frac{z^2}{2}\right)(-\mathtt{Y})\Gamma\left(\frac{1}{2}\,(\mathtt{I}-\mathtt{Y})\right)^{-1}&\\ + \frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}{}_1F_1\left(\frac{1}{2}\,(\mathtt{I}-\mathtt{Y})\,; \frac{3}{2}\mathtt{I}; \frac{z^2}{2}\right)\Gamma\left(-\frac{1}{2}\mathtt{Y}\right)^{-1}\mathrm{sgn}(z)&\\ + \frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}{}_1F_1\left(\frac{1}{2}\,(3\mathtt{I}-\mathtt{Y})\,; \frac{5}{2}\mathtt{I}; \frac{z^2}{2}\right)\frac{1}{3}\,(\mathtt{Y}-\mathtt{I})\,\Gamma\left(-\frac{1}{2}\mathtt{Y}\right)^{-1}z^2&\end{aligned}$$

for $z \in \mathbb{R} \setminus \{0\}$. In $z = 0$ the function $\mathrm{F}_{\mathtt{Y}}(z)$ is not differentiable as we have

$$\begin{aligned}\partial_+\,\mathrm{F}_{\mathtt{Y}}(z)|_{z=0} &= \frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}\Gamma\left(-\frac{1}{2}\mathtt{Y}\right)^{-1},\\ \partial_-\,\mathrm{F}_{\mathtt{Y}}(z)|_{z=0} &= -\frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}\Gamma\left(-\frac{1}{2}\mathtt{Y}\right)^{-1}.\end{aligned}$$

Now we want to examine the asymptotic behaviour of the second Kummer function with matrix parameters. Following a standard approach from Slater [96, p. 35] or Paris and Kaminski [82, p. 106]), we compute the Mellin-Barnes integral using the residue theorem.

Lemma 22. *Let* $\mathtt{L}$ *be a positive stable, diagonalizable matrix. For* $z \in \mathbb{C}$ *with* $|\arg(z)| < \frac{3\pi}{2}$ *and* $c < \infty$ *holds*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(-s)\Gamma(\mathtt{L}+s\mathtt{I})\Gamma(\mathtt{L}+(1-b+s)\mathtt{I})|z|^{-s}ds \\ = |z|^{\mathtt{L}}\,\mathrm{U}(\mathtt{L}, b\mathtt{I}, z)\Gamma(\mathtt{L})\Gamma(\mathtt{L}+(1-b)\mathtt{I}).$$

Proof. With $\lambda_1,\ldots,\lambda_N$ we denote the eigenvalues of $\mathtt{L}$. Since $\mathtt{L}$ is diagonalizable, we have the eigenvalue decomposition $\mathtt{L} = \mathtt{T}\Lambda\mathtt{T}^{-1}$ where $\Lambda = \operatorname{diag}((\lambda_1,\ldots,\lambda_N)^{\mathsf{T}})$ and $\mathtt{T}$ is the matrix of the corresponding eigenvectors. We remark that $\mathtt{L}+k\mathtt{I}$ has the the eigenvalue decomposition $\mathtt{T}(\Lambda+k\mathtt{I})\mathtt{T}^{-1}$ for all $k \in \mathbb{N}$. As all eigenvalues have non-negative real part, the singularities of $\Gamma(\lambda_i + s)$ are on the negative real axis. Let α, β and R be positive numbers. For each eigenvalue, we define an integral

$$\mathfrak{I}_{\lambda_i,R} = \frac{1}{2\pi i}\oint_{\mathcal{C}_{\lambda_i}} \Gamma(-s)\Gamma(\lambda_i+s\mathtt{I})\Gamma(\lambda_i+(1-b+s))|z|^{-s}ds,$$

where the curve $\mathcal{C}_{\lambda_i}$ is taken around a rectangular contour so that the poles at $s = -\lambda_i - k$ and at $s = -\lambda_i - (1-b+k)$ are inside the contour for all $i \in 1,\ldots,N$ and $k = 0,1,2,\ldots,\lfloor R\rfloor$. The poles of $\Gamma(-s)$ are outside this contour. The residue theorem gives us

$$\mathfrak{I}_{\lambda_i,R} = |z|^{\lambda_i}\sum_{k=0}^{\lfloor R\rfloor}\Gamma(\lambda_i+k)\Gamma(1-b-k)\frac{(-|z|)^k}{k!} \\ + |z|^{\lambda_i}|z|^{1-b}\sum_{k=0}^{\lfloor R\rfloor}\Gamma(\lambda_i+1-b+k)\Gamma(b-(k+1))\frac{(-|z|)^k}{k!}.$$

Now we define a matrix

$$\mathfrak{I}_R = \mathtt{T}\operatorname{diag}((\mathfrak{I}_{\lambda_1,R},\ldots,\mathfrak{I}_{\lambda_N,R})^{\mathsf{T}})\,\mathtt{T}^{-1}.$$

Using the fact $\mathtt{T}\operatorname{diag}((\Gamma(\lambda_1),\ldots,\Gamma(\lambda_1))^{\mathsf{T}})\,\mathtt{T}^{-1} = \Gamma(\mathtt{L})$, we obtain

$$\mathfrak{I}_R = |z|^{\mathtt{L}}\sum_{k=0}^{\lfloor R\rfloor}\Gamma(\mathtt{L}+k\mathtt{I})\Gamma(1-b-k)\frac{(-|z|)^k}{k!} \\ + |z|^{\mathtt{L}}|z|^{1-b}\sum_{k=0}^{\lfloor R\rfloor}\Gamma(\mathtt{L}+(1-b+k)\mathtt{I})\Gamma(b-(k+1))\frac{(-|z|)^k}{k!}.$$

Using $\Gamma(\mathtt{L}+k\mathtt{I}) = (\mathtt{L})_k\Gamma(\mathtt{L})$ for Gamma matrix functions and

$$(-1)^k\Gamma(1-b-k) = \frac{\Gamma(1-b)}{(b)_k}$$

for Gamma functions and writing $\mathfrak{I} = \lim_{R\to\infty}\mathfrak{I}_R$, we get

$$\begin{aligned}\mathfrak{I} &= |z|^{\mathtt{L}}\sum_{k=0}^{\infty}\frac{(\mathtt{L})_k}{(b)_k}\frac{|z|^k}{k!}\Gamma(\mathtt{L})\Gamma(1-b) \\ &\quad + |z|^{\mathtt{L}}|z|^{1-b}\sum_{k=0}^{\infty}\frac{(\mathtt{L}+(1-b)\mathtt{I})_k}{(2-b)_k}\frac{|z|^k}{k!}\Gamma(\mathtt{L}+(1-b)\mathtt{I})\Gamma(b-1) \\ &= |z|^{\mathtt{L}}{}_1F_1(\mathtt{L}; b\mathtt{I}; z)\,\Gamma(\mathtt{L})\Gamma(1-b) \\ &\quad + |z|^{\mathtt{L}}|z|^{1-b}{}_1F_1(\mathtt{L}+(1-b)\mathtt{I}; (2-b)\mathtt{I}; z)\,\Gamma(\mathtt{L}+(1-b)\mathtt{I})\Gamma(b-1).\end{aligned}$$

Since $\mathtt{L}$ and $\mathtt{L}+(1-b)\mathtt{I}$ commute, also their Gamma matrix functions commute. Therefore, we get

$$\mathfrak{I}=|z|^{\mathtt{L}}\,\mathtt{U}(\mathtt{L},b\mathtt{I},z)\Gamma(\mathtt{L})\Gamma(\mathtt{L}+(1-b)\mathtt{I}).$$

In the next step, we find an integral representation for $\mathfrak{I}$. Therefore, we examine the contour $\mathcal{C}_{\lambda_i}$ for $R\to\infty$ for each eigenvalue λ_i. We introduce the abbreviations

$$\begin{aligned}
\mathfrak{I}_{\lambda_i,\mathcal{C}_1}&=\frac{1}{2\pi i}\int_c^{-R}\Gamma(-x-i\alpha)\Gamma(\lambda_i+x+i\alpha)\Gamma(\lambda_i+1-b+x+i\alpha)|z|^{-x-i\alpha}dx,\\
\mathfrak{I}_{\lambda_i,\mathcal{C}_2}&=-\frac{1}{2\pi i}\int_{c-i\beta}^{c+i\alpha}\Gamma(R-t)\Gamma(\lambda_i-R+t)\Gamma(\lambda_i+1-b-R+t)|z|^{R-t}dt,\\
\mathfrak{I}_{\lambda_i,\mathcal{C}_3}&=\frac{1}{2\pi i}\int_{-R}^{c}\Gamma(-x+i\beta)\Gamma(\lambda_i+x-i\beta)\Gamma(\lambda_i+1-b+x-i\beta)|z|^{-x+i\beta}dx,\\
\mathfrak{I}_{\lambda_i,\mathcal{C}_4}&=\frac{1}{2\pi i}\int_{c-i\beta}^{c+i\alpha}\Gamma(-s)\Gamma(\lambda_i+s)\Gamma(\lambda_i+1-b+s)|z|^{-s}ds.
\end{aligned}$$

Parametrising the contour, we obtain

$$\mathfrak{I}_{\lambda_i,R}=\mathfrak{I}_{\lambda_i,\mathcal{C}_1}+\mathfrak{I}_{\lambda_i,\mathcal{C}_2}+\mathfrak{I}_{\lambda_i,\mathcal{C}_3}+\mathfrak{I}_{\lambda_i,\mathcal{C}_4}.$$

Now, we show that $\mathfrak{I}_{\lambda_i,\mathcal{C}_1}\to 0$ and $\mathfrak{I}_{\lambda_i,\mathcal{C}_3}\to 0$ in a similar way as [96]. The Stirling formula

$$|\Gamma(u+iv)|\lesssim\sqrt{2\pi}|v|^{u-\frac{1}{2}}e^{-\frac{\pi}{2}|v|}$$

for the Gamma function with $|u|$ finite and $|v|$ large (see [26, p. 223]) gives us the asymptotic inequalities

$$\begin{aligned}
|\Gamma(-x-i\alpha)|&\lesssim\sqrt{2\pi}\alpha^{-x-\frac{1}{2}}e^{-\frac{\pi}{2}\alpha},\\
|\Gamma(\lambda_i+x+i\alpha)|&\lesssim\sqrt{2\pi}\,(\alpha+\mathrm{Im}\,(\lambda_i))^{\mathrm{Re}\,(\lambda_i)+x-\frac{1}{2}}\,e^{-\frac{\pi}{2}(\alpha+\mathrm{Im}\,(\lambda_i))},\\
|\Gamma(\lambda_i+1-b+x+i\alpha)|&\lesssim\sqrt{2\pi}\,(\alpha+\mathrm{Im}\,(\lambda_i))^{\mathrm{Re}\,(\lambda_i)+1-b+x-\frac{1}{2}}\,e^{-\frac{\pi}{2}(\alpha+\mathrm{Im}\,(\lambda_i))}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\mathfrak{I}_{\mathcal{C}_1}|\leq\sqrt{2\pi}\int_c^{-R}&|\Gamma(-x+i\alpha)|\,|\Gamma(\lambda_i+x-i\alpha)|\\
&|\Gamma(\lambda_i+1-b+x-i\alpha)|\,|z|^{-x}e^{\alpha\arg(z)}dx\\
\lesssim\sqrt{2\pi}\int_c^{-R}&\alpha^{-x-\frac{1}{2}}\,(\alpha+\mathrm{Im}\,(\lambda_i))^{2\,\mathrm{Re}\,(\lambda_i)+2x-b}\\
&e^{-\pi\,\mathrm{Im}\,(\lambda_i)}|z|^{-x}e^{-\alpha\left(\frac{3\pi}{2}-\arg(z)\right)}dx
\end{aligned}$$

and, since $\arg(z)<\frac{3\pi}{2}$, we obtain

$$\lim_{\alpha\to\infty}|\mathfrak{I}_{\lambda_i,\mathcal{C}_1}|=0.$$

Almost analogously, we find that

$$\lim_{\beta\to\infty}|\mathfrak{I}_{\lambda_i,\mathcal{C}_3}|=0.$$

In the last step, we analyse $\mathfrak{I}_{\lambda_i,\mathcal{C}_2}$. We define

$$\mathfrak{I}_{\lambda_i,R}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(R-t)\Gamma(\lambda_i-R+t)\Gamma(\lambda_i+1-b-R+t)|z|^{R-t}dt.$$

Since t is complex valued, a slightly different calculation using the Stirling formula and $|\Gamma(z)| \leq |\Gamma(\mathrm{Re}\,(z))|$, we obtain

$$\lim_{R\to\infty} |\mathfrak{I}_R| = 0.$$

Finally, we get

$$\begin{aligned}\mathfrak{I} &= \lim_{R\to\infty} \mathtt{T}\,\mathrm{diag}\,((\mathfrak{I}_{\lambda_1,R},\dots,\mathfrak{I}_{\lambda_N,R})^{\mathsf{T}})\,\mathtt{T}^{-1} = \mathtt{T}\,\mathrm{diag}\,((\mathfrak{I}_{\lambda_1,\mathcal{C}_4},\dots,\mathfrak{I}_{\lambda_N,\mathcal{C}_4})^{\mathsf{T}})\,\mathtt{T}^{-1}\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(-s)\Gamma(\mathtt{L}+s\mathtt{I})\Gamma(\mathtt{L}+(1-b+s)\mathtt{I})|z|^{-s}ds.\end{aligned}$$

Combing this with the result from the residue theorem completes the proof. □

In the following theorem, we take a closer look at the asymptotic behaviour whenever $|z| \to \infty$.

Theorem 19. *Let* $\mathtt{L}$ *be a diagonalisable matrix. For* $|z| \to \infty$ *the second Kummer function behaves as*

$$\mathrm{U}(\mathtt{L}, b\mathtt{I}, z) \sim |z|^{-\mathtt{L}}.$$

Proof. Fix $R > 0$. Similar to [96, p. 58], we define a contour integral

$$\mathfrak{I}_R = \frac{1}{2\pi i}\oint_{\mathcal{C}_R^+} \Gamma(-s)\Gamma(\mathtt{L}+s\mathtt{I})\Gamma(\mathtt{L}+(1-b+s)\mathtt{I})|z|^{-s}ds$$

where the curve $\mathcal{C}_R^+$ is constructed such that all poles of $\Gamma(-s)$ lie inside and the poles of $\Gamma(\mathtt{L}+s\mathtt{I})$ and $\Gamma(\mathtt{L}+(1-b+s)\mathtt{I})$ outside. This is possible, because examining the residues of the Gamma matrix function, we get

$$\begin{aligned}\Gamma(\mathtt{L}+s\mathtt{I}) &= \int_0^\infty e^{-t}t^{\mathtt{L}+s\mathtt{I}-\mathtt{I}}dt\\ &= \int_0^1 \sum_{k=0}^\infty t^{\mathtt{L}+(s+k-1)\mathtt{I}}\frac{(-1)^k}{k!}dt + \int_1^\infty e^{-t}t^{\mathtt{L}+s\mathtt{I}-\mathtt{I}}dt\\ &= \sum_{k=0}^\infty \int_0^1 e^{(\mathtt{L}+s\mathtt{I}+(k-1)\mathtt{I})\ln(t)}\frac{(-1)^k}{k!}dt + \int_1^\infty e^{-t}t^{\mathtt{L}+s\mathtt{I}-\mathtt{I}}dt\\ &= \sum_{k=0}^\infty \left(\int_{-\infty}^0 e^{(\mathtt{L}+s\mathtt{I}+k\mathtt{I})u}du\right)\frac{(-1)^k}{k!} + \int_1^\infty e^{-t}t^{\mathtt{L}+s\mathtt{I}-\mathtt{I}}dt\\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!}(\mathtt{L}+s\mathtt{I}+k\mathtt{I})^{-1} + \int_1^\infty e^{-t}t^{\mathtt{L}+s\mathtt{I}-\mathtt{I}}dt.\end{aligned}$$

Obviously, $\Gamma(\mathtt{L}+s\mathtt{I})$ has simple poles whenever $\det(\mathtt{L}+(s+k)\mathtt{I}) = 0$ for $k \in \mathbb{N}$. Hence, $\Gamma(\mathtt{L}+s\mathtt{I})$ is singular if $s = -\lambda_i - k$ for all eigenvalues $\lambda_1,\dots,\lambda_N$ of $\mathtt{L}$. As $|\mathfrak{I}_R|$ is bounded for large $|z|$, we can write

$$\mathfrak{I} = \lim_{R\to\infty}\mathfrak{I}_R = \left(\sum_{k=0}^\infty \Gamma(\mathtt{L}+k\mathtt{I})\Gamma(1-b-k)\frac{(-|z|)^{-k}}{k!}\right)$$

by the residue theorem. Obviously, for $|z| \to \infty$, the integral $\mathfrak{I}$ converges to the unit matrix. On the other hand we already know from Lemma 22 that

$$\mathfrak{I} = |z|^{\mathtt{L}}\,\mathrm{U}(\mathtt{L}, b\mathtt{I}, z)\Gamma(\mathtt{L})\Gamma(\mathtt{L}+(1-b)\mathtt{I}).$$

and, hence,

$$\mathrm{U}(\mathtt{L}, b\mathtt{I}, z) = |z|^{-\mathtt{L}}\mathfrak{I}\Gamma(\mathtt{L})^{-1}\Gamma(\mathtt{L} + (1-b)\mathtt{I})^{-1}.$$

□

From the eigenvalue decomposition $\mathtt{L} = \mathtt{T}\Lambda\mathtt{T}^{-1}$, we get

$$\lim_{c\to\infty} e^{-c\mathtt{L}} = \mathtt{T} \lim_{c\to\infty} \operatorname{diag}\left(\left(e^{-\lambda_1 c}, \ldots, e^{-\lambda_N c}\right)^{\mathsf{T}}\right) \mathtt{T}^{-1} = \mathtt{T}\mathtt{0}_{N\times N}\mathtt{T}^{-1} = \mathtt{0}_{N\times N},$$

where $\mathtt{0}_{N\times N}$ is an $N \times N$ matrix with zero in all entries. If $\mathtt{L}$ is positive stable and diagonalizable, Theorem 19 give us $\lim_{x\to\infty} \mathrm{U}(\mathtt{L}, b\mathtt{I}, x) = \mathtt{0}_{N\times N}$ for $x \in \mathbb{R}$. Hence,

$$\lim_{x\to\infty} \mathrm{F}_{\mathtt{K}}(x) = \lim_{x\to\infty} \mathrm{U}\left(-\frac{1}{2}\mathtt{K}, \frac{1}{2}\mathtt{I}, \frac{x^2}{2}\right) = \mathtt{0}_{N\times N}$$

holds for negative stable matrices $\mathtt{K}$.

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