Rauzy fractals and tilings

Milton Minervino

DISSERTATION

Betreuer:
Ao.Univ.-Prof. Dipl.-Ing. Dr.techn. Jörg Thuswaldner

Lehrstuhl für Mathematik und Statistik
Department Mathematik und Informationstechnologie
Montanuniversität Leoben

Leoben, im April 2014

This thesis was written within the framework of the

Doctoral Program "Discrete Mathematics"

supported by the Austrian Science Fund
(FWF grant W1230)

DOCTORAL PROGRAM
DISCRETE MATHEMATICS

TU & KFU GRAZ • MU LEOBEN
AUSTRIA

www.math.tugraz.at/discrete

Affidavit

I declare in lieu of oath, that I wrote this thesis and performed the associated research myself, using only literature cited in this volume.

....

date (signature)

Acknowledgements

This thesis is dedicated to the wonderful people I have met in Graz during my stay. The first I want to mention are of course the closest and best friends I have: my amazing beloved Maygasse flatmates Kathi, Margit, Marina, Sebastian, and my former ones Flo and Linda. I have no words to thank them for the intense and colourful moments we lived and shared together. You are my Austrian family and you cannot imagine how important for me you have become.

A huge acknowledgement is for the beautiful people with whom I played music during these years, who made me smile, feel good, dance and swing softly in delightful soundscapes. These are in particular the Marinski Combo and Vento Sul. Fascinating personalities and exquisite musicians. Without them something would have missed incredibly inside me, you made me extremely happy. A special thank is for Regine and Vero, for the crazy deep emotional theatre experience we had together, so deep and unforgettable.

A sincere gratitude goes to my supervisor Jörg Thuswaldner, for having me constantly supported and motivated with many fruitful mathematical discussions, for the encouragement he could give me when things seemed incredibly complicated, and for his patience. I thank warmly the friendly Leoben team for the kind help and pleasant working environment, including the lunch and coffee time spent together. I praise our Doctoral Program in Discrete Mathematics funded by the Austrian Science Fund (FWF) for the support which permitted me to travel worldwide for conferences, academic visits and research periods abroad, and especially for the great international environment it created, not only in terms of research. A special thank goes to all my Ph.D. colleagues for the very good time spent together. I am grateful to the Paris team and especially to Valérie Berthé, Wolfgang Steiner and Timo Jolivet for the nice and funny experience I had in Paris.

And then there are all the others, many many people, among which my parents, for their love and the surprise packages and visits. But all of you must know that you played an important and precious role in my life. Without you I would not be what I am.

Contents

Introduction

This thesis is about Rauzy fractals, geometric objects arising in the study of symbolic dynamical systems generated by a particular class of substitutions. One of the main dynamical problems in this field is translated geometrically to a tiling problem by Rauzy fractals. We start with an overview of this famous open problem, giving special emphasis to the geometric interpretation. The second part describes the principal issues when going beyond the main hypotheses. This is indeed the main subject of this thesis. The third part sketches the contributions and advances we have made, which will be described in full detail in the subsequent chapters.

The Pisot conjecture

Substitutions, dynamical systems and tilings of the line. *Substitutions* are simple combinatorial objects which replace letters of a finite alphabet by finite words. They generate infinite words by iteration which can be seen geometrically as tilings of the line by associating a length with each letter. A substitution is intrinsically a self-similar object since it inflates each tile and subdivides it into translates of the original tiles. Given a primitive substitution σ we can consider the set X_σ of bi-infinite words having the same language as a bi-infinite periodic point of σ and consider the $\mathbb{Z}$-action of the shift S on X_σ. The symbolic dynamical system (X_σ, S) is called *substitution dynamical system*, or *substitutive system*. On the other side we can define a tiling metric in which two tilings of the line are close if they agree up to a small translation in a large neighbourhood of the origin. Given a tiling $\mathcal{T}$ of the line we can consider the closure with respect to the tiling metric $X_\mathcal{T} = \overline{\{\mathcal{T} - t : t \in \mathbb{R}\}}$. The $\mathbb{R}$-action by translations on $X_\mathcal{T}$ is called the *tiling flow*, denoted $(X_\mathcal{T}, \mathbb{R})$. The symbolic and tiling points of view are of course related. The substitution dynamical systems (X_σ, S) are cross-sections of tiling flows. More precisely the system $(X_\mathcal{T}, \mathbb{R})$ is topologically conjugate to the suspension of (X_σ, S) with roof function given by a vector of lengths associated with the letters, usually chosen to be a left eigenvector of the incidence matrix M_σ of the substitution associated with the Perron-Frobenius eigenvalue. One-dimensional substitution dynamical systems are minimal, uniquely ergodic with zero entropy. Similar considerations hold for tiling flows. It is natural to investigate further the ergodic behaviour of these systems.

The spectral type of one-dimensional substitution dynamical systems can vary from the weakly mixing one to the one with pure discrete spectrum, depending on the substitution. Recall that a measure-preserving dynamical system (X, T, μ) has *pure discrete spectrum* if the eigenfunctions span a dense subspace of $L^2(X, \mu)$. Dekking [Dek78] analysed the case of substitutions of constant length and gave a characterization for the discreteness of the spectrum in connection with a notion of coincidence. For substitutions of non-constant length a result of [DK78] established that strongly mixing substitutive systems do not exist. It is a consequence of the work of Host [Hos86] that all eigenfunctions of primitive substitutive dynamical systems are continuous and that the spectrum of a substitutive system can be split into two parts. The first part is of arithmetic origin, and depends only on the incidence matrix of the substitution. The second part has a combinatorial origin, and is related to the return words of the fixed point of the substitution (see [FMN96]). For a general and detailed overview of the spectral theory of substitutive systems, we refer to [Fog02, Chapter 7] and [Que10].

Kronecker systems, i.e. rotations on compact Abelian groups, are the canonical examples of measure-preserving transformations with discrete spectrum. By a theorem of Halmos and Von Neumann, a measure-preserving transformation with discrete spectrum is metrically isomorphic to a Kronecker system (see [Wal82, CFS82, EW11]).

The conjecture. The following is known as the *Pisot conjecture*.

CONJECTURE. *Let σ be an irreducible unit Pisot substitution. Then (X_σ, S) has pure discrete spectrum.*

The importance of the hypotheses *irreducible, unit* and *Pisot* must be underlined. For definitions see Chapter 1. We will see in the second part of the introduction what happens when we leave the framework given by the first two hypotheses.

The same conjecture can be formulated for tiling flows and indeed it is equivalent to consider substitution dynamical systems or tiling flows since it was proven in [CS03] that for irreducible Pisot substitutions the tiling flow has pure discrete spectrum if and only if the substitutive system does. We will privilege in this thesis the symbolic approach. For more on the topology of tiling spaces we refer to [Sad08].

Why Pisot? The Pisot assumption in this theory is fundamental. We exhibit dynamical and geometrical reasons for the importance of this assumption.

The role of Pisot numbers in the study of mathematical quasicrystals was already pointed out in [BT87]. Lind [Lin84] and Thurston [Thu89] showed that *Perron numbers*, that is, algebraic integers $\lambda > 1$ whose Galois conjugates are in modulus strictly less than λ, can be the only expansion factors for self-affine tilings of $\mathbb{R}$. In [Sol97, Sol07] a complete characterization of the eigenvalues for tiling flows was carried out. In particular, using this characterization and the classical theorem of Pisot, it was shown that the tiling flow has non-trivial eigenvalues, equivalently is not weakly mixing, if and only if the inflation factor is a Pisot number.

The importance of the Pisot hypothesis is highlighted also by the geometric representation, which will lead to *Rauzy fractals*. The action of the incidence matrix M_σ of an irreducible unit Pisot substitution on $\mathbb{R}^n$, where n is the number of letters on which σ acts, gives an expanding/contracting (or unstable/stable) M_σ-invariant decomposition $\mathbb{R}^n = \mathbb{K}^e_\beta \oplus \mathbb{K}^c_\beta$. The action of M_σ restricted to $\mathbb{K}^e_\beta \cong \mathbb{R}$ is a dilation by the Pisot number $\beta > 1$, while it is a contraction on $\mathbb{K}_\beta^c \cong \mathbb{R}^{n-1}$ by $|\beta'| < 1$ for all β' Galois conjugates of β. This remarkable dynamical property of Pisot numbers will be crucial: the contracting space $\mathbb{K}_\beta^c$ will be suitable to represent geometrically the substitution dynamical system by a fractal attractor generated by a graph directed iterated function system with contraction factors given by the Galois conjugates of β. Good references on fractal geometry are [Fal03, Bar88].

Origins of the geometric interpretation. The geometric theory for the study of substitution dynamical systems was initiated by Gérard Rauzy in his seminal work [Rau82]. He succeeded to prove that the substitution dynamical system (X_σ, S) generated by the *Tribonacci substitution* $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ is a translation on a two-dimensional torus. The key point was to interpret the shift as a domain exchange on a fractal domain, later called Rauzy fractal in his honour, decomposable in three subpieces, or subtiles, which give a suitable partition for the domain exchange to be coded by (X_σ, S). Another essential point is that Rauzy showed also that the fractal domain obtained with this construction can tile periodically the plane where it is represented. Therefore this domain can be seen as a two-dimensional torus and the domain exchange as a translation on this torus.

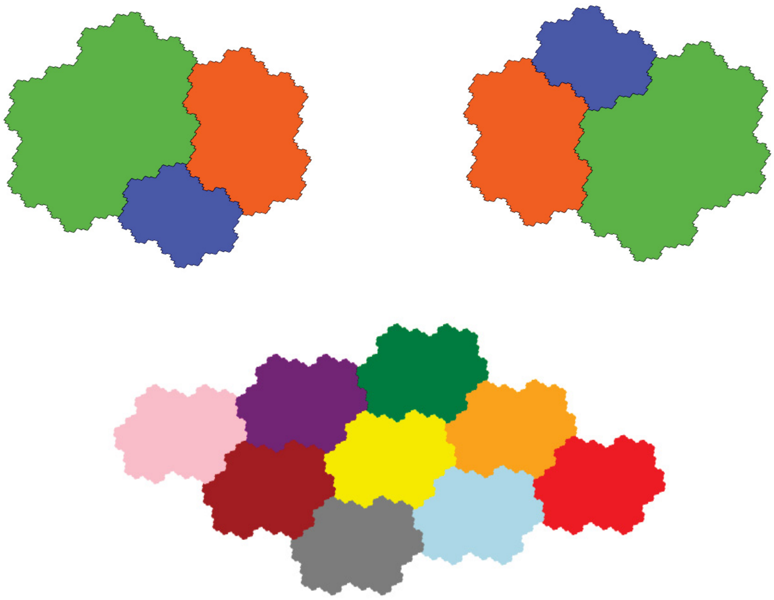


FIGURE 1. Domain exchange and periodic tiling for the Rauzy fractal associated with the Tribonacci substitution.

Rauzy's original idea was to use a special kind of numeration with admissibility governed by a graph associated with the substitution to obtain the fractal domain as geometrical representation of the substitutive system. We will see in Chapter 1 how substitutions and numeration are intimately related.

Beta-numeration is a particular case of the substitutive one and there is an extensive and independent study focused on it. The investigation of tilings generated by beta-numeration began with the groundwork of Thurston [Thu89] who, inspired by Rauzy, produced Euclidean self-similar tilings as geometrical picture of the expansion of numbers in a Pisot unit base β. Rauzy fractals were obtained by embedding the beta-integers, i.e. the elements whose β-expansions have only non-negative powers of β, via a Minkowski embedding into $\mathbb{K}_\beta^c$.

We will see in the sequel that Rauzy fractals can be defined in several equivalent ways besides the numeration system style. However, numeration associated with substitutions and beta-numeration will be our main point of view in Chapter 2 and Chapter 3.

A quest of tilings. Rauzy's construction was extended in [AI01, CS01b] to every irreducible unit Pisot substitution satisfying a certain combinatorial property, called the *strong coincidence condition*. This condition, which is true for substitutions on two letters [BD02] and is conjectured to be true for every irreducible Pisot substitution, is sufficient to get the measure-disjointness of the subpieces of the Rauzy fractal and thus allows to define a domain exchange on it. It turns out that the shift on the substitutive system is measurably conjugate to the domain exchange and it is semi-conjugate and almost everywhere m-to-one to a toral translation, with m constant. This last observation is motivated by the fact that the Rauzy fractals induce a *periodic multiple tiling* with covering degree m when translated by a suitable lattice associated with the domain exchange transformation. The main point is to get a perfect tiling ($m = 1$). Then the substitutive system is conjugate to a toral translation, which implies the pure discreteness of the spectrum. In this way the Pisot conjecture has been translated to a tiling problem.

[AI01] and [SW02] established many elementary properties of Rauzy fractals, among them the important fact that they satisfy a set equation governed by the so-called prefix-suffix graph of the substitution. Basic topological properties for Rauzy fractals read as follows (see e.g. [BST10]):

(1) They are compact sets with non-empty interior.
(2) They are the closure of their interior.
(3) Their fractal boundary has measure zero.

More topological properties like connectedness, homeomorphy to a disk [ST09] and considerations on the fundamental group [JLL13] were recently studied.

A major breakthrough was made in [AI01] with the introduction of *geometric realizations of substitutions and their duals*. Fixed points of substitutions over the alphabet $\mathcal{A} = \{1, 2, \dots, n\}$ can be seen geometrically as "broken lines" in $\mathbb{R}^n$ made of translates of segments parallel to the basis vectors $\mathbf{e}_a$, $a \in \mathcal{A}$. Rauzy fractals can be seen as the closure of the projection of the vertices of the broken line into the contracting space $\mathbb{K}_\beta^c$ along the expanding direction $\mathbb{K}_\beta^e$. A certain operator $\mathbf{E}_1(\sigma)$ is the geometric realization of the substitution on segments and broken lines. We can consider the dual map $\mathbf{E}_1^*(\sigma)$ and interpret it as a map on faces of codimension one. Duals of substitutions have been used in connection to stepped surfaces, which play a central role in [AI01] in the context of irreducible substitutions.

Stepped surfaces were first defined in [Rev91] and used as arithmetic discrete models for hyperplanes for example in [IO93, IO94]. A stepped surface is seen in [AI01] as the set of nearest coloured points of $\mathbb{Z}^n$ above the contracting space $\mathbb{K}_\beta^c$ of the substitution σ (see also [IR06] for a good detailed description). The

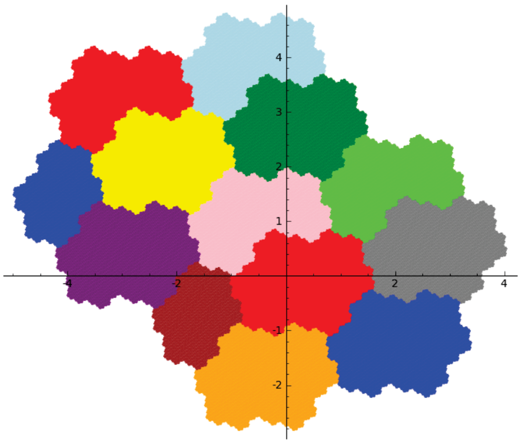
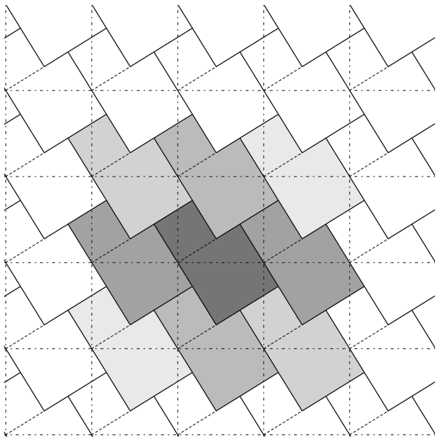


FIGURE 2. Patch of the self-replicating tiling made of Rauzy fractals for the Tribonacci substitution.

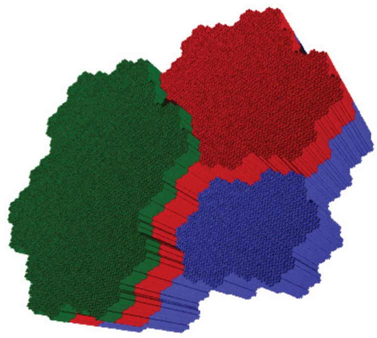
projection of these points into $\mathbb{K}_\beta^c$ forms a discrete aperiodic set, in particular a Delone set. To any coloured point of the stepped surface one can associate a hypercube face of a certain type. The union of faces approximating the contracting representation space of the substitution is called a *geometrical representation of the stepped surface* and it is invariant under the dual $\mathbf{E}_1^*(\sigma)$. The projection of the stepped surface into $\mathbb{K}_\beta^c$ is a polygonal tiling. If we replace these polygons by Rauzy fractals we obtain a *self-replicating multiple tiling*, in the sense that the $\mathbf{E}_1^*(\sigma)$-invariance gives the self-replicating property defined by Kenyon [Ken92]. Furthermore using the dual formalism we can define Rauzy fractals as Hausdorff limits of renormalized iterations under the dual geometric substitution $\mathbf{E}_1^*(\sigma)$ of faces.

Aperiodic tilings (among which we mention the celebrated Penrose tiling) serve as mathematical models of atomic configurations for *quasicrystals* [Sen95, BG13]. Physical quasicrystals are metallic alloys which exhibit sharp bright spots, called Bragg peaks, as point-like as those of crystals in their X-ray diffraction pattern, but have aperiodic structure, usually manifested by the presence of a non-crystallographic symmetry. They were discovered in 1982 by Dan Shechtman, who subsequently won the Nobel prize in 2011, and they revolutionised this field since lattice symmetry, crystal structure, and pure point diffraction were considered as synonymous. A strong motivation for questions on pure discrete spectrum of tiling flows and substitution dynamical systems comes from the equivalence with pure point diffractivity of atomic structures [LMS02].

Rauzy fractals induce a third kind of tiling related to *Markov partitions* for hyperbolic toral automorphisms. A partition of the underlying set of a dynamical system induces a coding of the orbits and hence a semiconjugacy with a subshift. Markov partitions are a special class of partitions for which the target is a subshift of finite type. For more details see e.g. [LM95, KH95, Adl98, BS02]. Markov partitions exist for every hyperbolic toral automorphism [Sin68, Bow70] and they can be constructed explicitly with two rectangles for square matrices of size two [AW70]. In higher dimensions no explicit construction is known and by a result of Bowen [Bow78] such partitions must have fractal boundary. We can suspend the subtiles of a Rauzy fractal with intervals of different lengths along the expanding direction and translate this suspended domain by $\mathbb{Z}^n$. In this way we obtain an explicit geometrical construction of a Markov partition for the Pisot toral automorphism M_σ associated with the substitution, provided that the suspended Rauzy fractal tiles $\mathbb{R}^n$ periodically. This was done in the irreducible unit case in [Pra99]. Purely periodic beta-expansions were characterized using this domain in [HI97, IR05, BS07]. Another approach based on homoclinic points appears in [ES97, Sch00] and one based on generalised radix representations with a matrix as base in [LB95]. We mention [KV98] for an arithmetic construction of sofic partitions of hyperbolic toral automorphisms beyond the Pisot case, and [AFHI11] for a Rauzy fractals construction of a Markov partition for a free group automorphism associated with a complex Pisot root. See also [LS05] for a treatment on non-expansive group automorphisms and the study of a two-sided beta-shift arising from a Salem number.



(A) Periodic tiling induced by the Markov partition for the Fibonacci automorphism $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$.



(B) Markov partition for the Tribonacci automorphism $\begin{pmatrix}1&1&1\\1&0&0\\0&1&0\end{pmatrix}$.

So far we have seen that an irreducible unit Pisot substitution induces the following multiple tilings:

- Periodic associated with a domain exchange.
- Aperiodic self-replicating associated with a stepped surface.
- Periodic related to a Markov partition for M_σ.

An important result of [IR06] asserts that these three collections are simultaneously tilings provided one of them is a tiling.

Raužy fractals have become an extremely important tool in the study of one-dimensional substitutive systems and of the Pisot conjecture. Furthermore a vast literature on their combinatorial, topological, dynamical, arithmetical and number-theoretical properties, besides the applications involving them in discrete geometry, automata, tilings and quasicrystals theory, has been flourishing.

State of the art. We present a non-exhaustive list of results, sufficient and necessary conditions for the Pisot conjecture.

- The Pisot conjecture is true for any substitution over two letters (see [HS03]).

- *Property (F)* was first introduced for beta-numeration in [FS92] and asserts that every non-negative element of $\mathbb{Z}[\beta^{-1}]$ has a finite beta-expansion. It is equivalent to the topological property that $\mathbf{0}$ is an exclusive inner point of the central tile. Thus it is a sufficient (but not necessary) condition for the tiling property. It can be stated geometrically by saying that iterating $\mathbf{E}_1^*(\sigma)$ on the initial patch of faces centred at $\mathbf{0}$ we obtain the whole stepped surface. It will be considered in Chapter 2.
- *Property (W)* is an arithmetical property introduced in the context of beta-numeration (see e.g. [Aki02, ARS04] and Chapter 3 for a precise definition and for classes of Pisot numbers satisfying it). It is equivalent to the tiling property.
- The *super coincidence condition* is a purely combinatorial condition introduced in [IR06] equivalent to the tiling property. Two segments have the same height if the intersection of the interiors of their projections on the expanding line is non-empty. They have coincidence if a positive iterate of $\mathbf{E}_1(\sigma)$ on the two segments has at least one segment in common. The super coincidence condition asserts that any two segments have a coincidence whenever they have the same height.
- The *geometric coincidence condition* was introduced in [BK06] and it is the analogous of the super coincidence condition in the tiling flow setting. The continuous map factoring the tiling flow onto its maximal equicontinuous factor, i.e. the Kronecker flow on a torus, is often called in the literature *geometric realization*. The geometric realization is non-trivial if and only if the substitution associated with the tiling flow is Pisot, and in this case it is almost everywhere m-to-1 for some positive integer m called *coincidence rank*. It is almost everywhere one-to-one if and only if the geometric coincidence condition holds, which is equivalent to the pure discreteness of the spectrum.
- *Boundary and contact graphs conditions*. The boundary graph describes the neighbouring tiles of an arbitrary tile in the self-replicating or in the periodic multiple tiling. The contact graph, introduced in the substitution settings in [Thu06], is based on polyhedral approximations of the Rauzy fractals and has a simpler construction and shape than the boundary graph (indeed it is the easier to compute). The tiling property is equivalent to the condition that the spectral radius of these graphs is less than the Pisot number β. The spectral condition on the boundary graph will be treated in Chapter 3.
- The *balanced pair algorithm* (known in another context as *overlap coincidence*) was introduced by Livshits [Liv87, Liv92] and is a purely combinatorial process which describes the growth of gaps between coincidence overlaps and checks whether these gaps are uniformly bounded. It terminates whenever the tiling property is satisfied (see [SS02] and [AL11] for advances).

In a very recent work [Bar14] it is shown that all beta-substitutions for β a Pisot simple Parry number have tiling flows with pure discrete spectrum, as do the

Pisot systems arising, for example, from the Jacobi-Perron and Brun continued fraction expansions. For the latter see also [BBJS14].

Good surveys on Rauzy fractals and the Pisot conjecture are [BS05, BST10, ABB+14].

Beyond unimodularity and irreducibility

We have seen that in the irreducible unit setting we can interpret the substitutive system geometrically as a domain exchange on the associated Rauzy fractal, which is represented in this case in a Euclidean space. The unimodularity and irreducibility assumptions play a prominent role.

When the Pisot number β is **non-unit**, i.e. $N(\beta) \neq \pm 1$, the guiding philosophy is to *enlarge algebraically the representation space in order to make β a unit*. This was conjectured already by Rauzy [Rau88] and requires to extend the representation space by certain $\mathfrak{p}$-adic factors. Precisely, we consider the locally compact subring

$$\mathbb{K}_\beta = \mathbb{K}_\infty \times \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p}$$

of the adèle ring $\mathbb{A}_{\mathbb{Q}(\beta)}$, consisting in the product $\mathbb{K}_\infty$ of the Archimedean completions of the number field $K = \mathbb{Q}(\beta)$ associated with the Galois embeddings together with the product of the non-Archimedean (or $\mathfrak{p}$-adic) completions $K_\mathfrak{p}$ determined by the prime divisors of the principal ideal (β) of the ring of integers $\mathcal{O}$ of K. By the product formula $\prod_\mathfrak{p} |\beta|_\mathfrak{p} = 1$ we have thus that β is a unit in $\mathbb{K}_\beta$. The representation space $\mathbb{K}_\beta$ has a hyperbolic decomposition into an expanding and a contracting space $\mathbb{K}_\beta^e \times \mathbb{K}_\beta^c$, where $\mathbb{K}_\beta^e \cong \mathbb{R}$ is the Archimedean completion associated with the identical Galois embedding. Indeed we call $\mathbb{K}_\beta^c$ the contracting space because $|\beta|_\mathfrak{p} < 1$ for every place $\mathfrak{p}$ occurring in $\mathbb{K}_\beta$ different from the identical Galois embedding. Furthermore multiplication by β is a uniform contraction in measure $\mu_c(\beta A) = \beta^{-1}\mu_c(A)$, for A measurable set, μ_c Haar measure on $\mathbb{K}_\beta^c$. The Rauzy fractals will be represented in $\mathbb{K}_\beta^c$. See Section 1.3 for a complete exposition on representation spaces.

Observe that in the non-unit case we have $M_\sigma \notin \mathrm{GL}_n(\mathbb{Z})$. The inverse of M_σ plays an important role in the definition of $\mathbf{E}_1^*(\sigma)$ and amounts to the inflation action in $\mathbb{K}_\beta^c$. If we considered the same purely Euclidean contracting space as in the unit case we would not be able to make geometric considerations because the Rauzy fractals would overlap in measure: the action of M_σ^{-1} on the stepped surface would not be invariant and would generate too many points. The choice of an enlarged representation space with $\mathfrak{p}$-adic factors permits to distribute the points of the stepped surface according to their $\mathfrak{p}$-adic height and to get finally again a discrete translation set.

In the non-unit realm the matrix M_σ can be seen as a Pisot solenoidal automorphism. A *solenoid* is a continuum, i.e. a compact connected topological space, that may be obtained as the inverse limit of continuous homomorphisms of topological groups. For example, given $(\mathbb{R}/\mathbb{Z}, T_2)$, where $T_2 : x \mapsto 2x \bmod 1$ is the circle-doubling map, the *dyadic solenoid* can be defined equivalently as

$$\varprojlim(\mathbb{R}/\mathbb{Z}, T_2) \cong (\mathbb{R} \times \mathbb{Q}_2)/\delta(\mathbb{Z}[\tfrac{1}{2}]) \cong \widehat{\mathbb{Z}[\tfrac{1}{2}]},$$

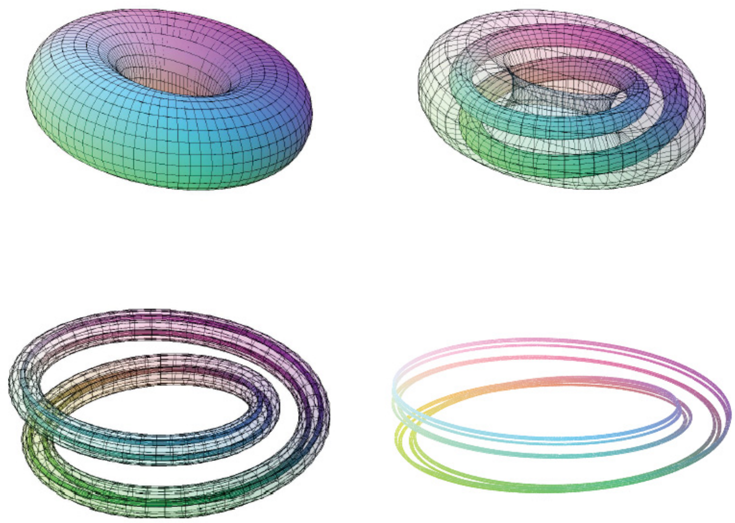


FIGURE 4. Dyadic solenoid seen as an attractor.

where δ is the diagonal embedding in $\mathbb{R} \times \mathbb{Q}_2$ and $\widehat{\mathbb{Z}[\frac{1}{2}]}$ is the Pontryagin dual, that is, the group of continuous characters of $\mathbb{Z}[\frac{1}{2}]$. In general the dual group of a solenoid is a subgroup of $\mathbb{Q}^m$, for some $m \geq 1$. For more on solenoids and applications in dynamical systems see e.g. [Sma67, Wil74, LW88, CEW97, VS08].

In analogy with the example above we call $\mathbb{K}_\beta/\delta(\mathbb{Z}[\beta^{-1}])$ a *beta-solenoid* and we consider the dynamical system $([0,1), T_\beta)$, where $T_\beta : x \mapsto \beta x \bmod 1$ is the (greedy) beta-transformation. We will construct in Chapter 3 a natural extension for Pisot beta-numeration using Rauzy fractals. When the natural extension domain tiles periodically the representation space $\mathbb{K}_\beta$ modulo the lattice $\delta(\mathbb{Z}[\beta^{-1}])$ then it is a beta-solenoid and can be considered as a Markov partition for the Pisot solenoidal automorphism given by the incidence matrix M_σ of a beta-substitution.

Siegel [Sie03] defined for the first time Rauzy fractals for Pisot substitutions that are not necessarily unit. In his Ph.D. thesis, Sing [Sin06b] studied various properties of non-unit Rauzy fractals in the context of model sets. An upper and lower bound for the Hausdorff dimension of the boundary of these sets was given in [Sin06a]. In [BS07] real numbers having a purely periodic beta-expansion in a non-unit Pisot base β have been characterized using Rauzy fractals, and recently Akiyama et al. [ABBS08] investigated properties of non-unit Rauzy fractals in the context of beta-numeration, with special regard to the boundary graph and to the gamma function, a certain number-theoretical function related to purely periodic beta-expansions.

If the substitution is **reducible** we have an M_σ-invariant decomposition of $\mathbb{R}^n$ consisting in a hyperbolic space with related expanding/contracting splitting $\mathbb{K}_\beta^e \times \mathbb{K}_\beta^c$ of dimension $d = \deg(\beta)$ (as in the irreducible case) and an additional *supplementary* (or *neutral*) space $\mathbb{H}^s$. The standard procedure to obtain Rauzy fractals in this setting is to project the vertices of the broken line associated with the fixed point of the substitution into $\mathbb{K}_\beta^c$ along $\mathbb{K}_\beta^e \oplus \mathbb{H}^s$.

Many difficulties arise in the reducible case because the number of colours, and therefore the number of types of faces, is $n = \#\mathcal{A}$, which is greater than the dimension $d = \deg(\beta)$ of the representation space $\mathbb{K}_\beta$. As pointed out in [EIR06], one of the major problems is that the existence of a geometrical representation for stepped surfaces is unclear. In [EIR06] an abstract stepped surface is defined similarly as in the irreducible case as set of "nearest" coloured points and its invariance under the dual substitution $\mathbf{E}_1^*(\sigma)$ is shown. However no general concrete polygonal construction is given. A self-replicating collection made of Rauzy fractals and one related to Markov partitions of toral automorphisms are studied.

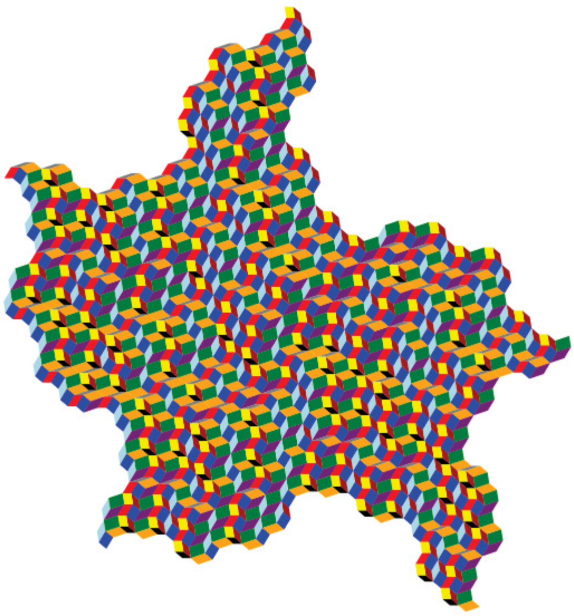


FIGURE 5. A projected patch of a stepped surface associated with a reducible substitution (see Chapter 4).

In [EI05] the authors found an ad hoc construction for a geometrical representation of the stepped surface of the Hokkaido substitution $\sigma : 1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$ related to the minimal Pisot number. Furthermore they proved that the substitution dynamical system (X_σ, S) is conjugate to a domain exchange on the Rauzy fractal but this cannot tile periodically. This curious phenomenon occurs in the reducible case and it is a significant difference compared to the irreducible setting. Further advances have been done in [ST09] where a quotient mapping condition is defined in order to have a periodic multiple tiling even in the reducible case. Nevertheless it is shown in [EI05] that for the Hokkaido substitution an extended domain satisfies the tiling property. This can be generally explained with the results of [BBK06]. They observed that for a wide class of beta-substitutions the domain exchange on the Rauzy fractal is the first return of a minimal toral translation on it. The extended fundamental domain is explained by taking into account the original Rauzy fractal plus the pieces prior to their first return.

The Pisot conjecture is not true for reducible Pisot substitutions. The Thue-Morse substitution $1 \mapsto 12, 2 \mapsto 21$ is a simple example of (constant length) reducible non-unit Pisot substitution. The associated substitution dynamical system is the coding of a skew-product of a dyadic rotation by $\mathbb{Z}/2\mathbb{Z}$ and has a non-discrete simple spectrum (see [Fog02, Chapter 5]). Combinatorially this substitution does not satisfy the strong coincidence condition, and geometrically one can see that the subtiles are overlapping in measure. Other counterexamples with non-constant length are provided e.g. in [BBK06]. It is not clear for which reducible substitutions the conjecture holds. We mention that no example of a beta-substitution failing the Pisot conjecture is known. Even the relation between the spectra of the substitutive system and of the tiling flow is not well understood. If the former has pure discrete spectrum then the latter does, but the opposite implication typically fails.

Recently irreducibility has been criticized as a natural assumption. Indeed one can take an irreducible Pisot substitution and rewrite it to obtain another substitution that is not irreducible but has topologically conjugate dynamics. In [BBJS12] a topological condition on the substitution is introduced: a Pisot substitution with Pisot number of degree d is called *homological Pisot* if the dimension of the first rational Čech cohomology of the tiling space is d. Recently reducible non-unit Pisot substitutions have gained importance because of the *coincidence rank conjecture*: if σ is a homological Pisot substitution with expansion β then the coincidence rank of σ divides $N(\beta)$. Advances in this direction have been obtained in [Bar13].

Contribution of this thesis

The aim of this thesis is to generalize to the non-unit, reducible case several dynamical, topological and arithmetical properties for the Rauzy fractals, including the tiling properties, and to investigate the main differences with the well-studied irreducible unit case. We extend to the non-unit case some important tiling conditions and equivalences, working with several different concepts of Rauzy fractals. We use new combinatorial and geometrical techniques to tackle the difficulties of the reducible case and we set up a new theory of Rauzy fractals generated by higher dimensional duals to better understand the dynamics of reducible Pisot substitutions.

Articles included in this thesis

[MT14]: **The geometry of non-unit Pisot substitutions**, with Jörg Thuswaldner, to appear in Annales de l'Institut Fourier (Grenoble), 64 (2014).

[MS14]: **Tilings for Pisot beta-numeration**, with Wolfgang Steiner, to appear in Indagationes Mathematicae (2014).

[Min14]: **Dynamics of reducible Pisot substitutions**, preprint.

In Chapter 1 we present all the necessary background notions that will be used throughout the thesis.

Chapter 2 is devoted to the results of [MT14]. We start defining the main objects of this thesis: *Rauzy fractals*. The beauty of these objects is that they appear naturally in various contexts. We present in Section 2.1 and 2.2 the following approaches:

(1) We review Dumont-Thomas numeration, which is a generalization of the well-known notion of beta-numeration, and view Rauzy fractals as the natural geometric objects related to this kind of numeration. This will be our main approach.

(2) We conceive the Rauzy fractals by projecting vertices of broken lines.

(3) We obtain the Rauzy fractals via a projective limit construction.

(4) We extend the geometric realization of a substitution and its dual studied in [AI01] to the non-unit case and define Rauzy fractals as renormalized pieces of stepped hypersurfaces with $\mathfrak{p}$-adic factors.

(5) We present Sing's [Sin06b] construction of Rauzy fractals via cut and project schemes and define them in terms of a graph directed iterated function system. In this framework Rauzy fractals occur as the dual prototiles of the multi-component model set associated with this cut and project scheme.

We show how these different approaches are related, provide conjugacies of the underlying mappings, and prove that they are all equivalent ways to view Rauzy fractals.

Particular importance is given to stepped surfaces, that is, coloured points of a certain lattice which are near in some sense to the contracting representation space $\mathbb{K}_\beta^c$. The projection of these points into $\mathbb{K}_\beta^c$ forms a Delone set which is a natural translation set for the Rauzy fractals. One of the main difficulties in the non-unit case was to give a geometrical representation to stepped surfaces and to see the Rauzy fractals as renormalized polygons under the dual of the geometric realization of the substitution. With our equivalent approaches we manage to give a concrete "shape" to the stepped surface and view the Rauzy fractals with this desired construction (see also [Sin06b]).

We establish geometric and topological properties of (non-unit) Rauzy fractals, some of which occur in Sing's thesis [Sin06b] in the context of model sets, some of them are new. In particular we prove that Rauzy fractals can be regarded as the solution of a graph directed set equation governed by the prefix graph of the substitution. This set equation provides a natural subdivision of the subtiles of a Rauzy fractal and highlights its self-affine structure that is inherited from the underlying substitution. We prove also basic topological properties like the equality to the closure of the interior and the zero measure for the boundary. These are the contents of Theorem 2.19. In Proposition 2.21 we discuss how Rauzy fractals are related to certain subshifts defined in terms of periodic points of the substitution σ and relate adic transformations to domain exchanges of subpieces of Rauzy fractals. In Theorem 2.23 we show that non-unit Rauzy fractals always admit a multiple tiling of the representation space $\mathbb{K}_\beta^c$. Moreover, extending results of [ABBS08] on non-unit beta-numeration we prove a tiling criterion for

Rauzy fractals. In particular, we show in Theorem 2.29 that Rauzy fractals admit a tiling of the representation space provided that the representations of the underlying Dumont-Thomas numeration obey a certain finiteness condition which is an extension of the well-known property (F) of beta-numeration (see [FS92]).

The results of [MS14] are presented in detail in Chapter 3, where the framework is Pisot *beta-numeration*, highlighting that we do not require the Pisot base to be a unit. We give an overview of this kind of numeration and the connection with Dumont-Thomas numeration in Section 1.2.2. Beta-numeration can be described by means of substitutions, precisely a particular class of them, called beta-substitutions. These have a fixed combinatorial structure and their incidence matrices correspond to the companion matrices of the polynomials associated with the substitutions. We note also that every beta-substitution satisfies the strong coincidence condition. Observe that beta-substitutions can be reducible and it is remarkable that no example of a beta-substitution failing the Pisot conjecture is known.

We discuss several objects: *Rauzy fractals*, *natural extensions*, and *integral beta-tiles*. We recall in Theorem 3.1 some of the main properties of Rauzy fractals associated with beta-numeration. It is well-known that they induce an aperiodic multiple tiling of their representation space, and there are several topological, combinatorial, and arithmetical conditions that imply the tiling property. In the irreducible unit context, having an aperiodic tiling is equivalent to having a periodic one [IR06]. The situation is different when we switch to the reducible and non-unit cases. In order to have a periodic tiling, a certain algebraic hypothesis (QM), first introduced in [ST09] for substitutions, must hold, and, when dealing with the non-unit case, our attention is naturally restricted to a certain *stripe space*, a subset of $\mathbb{K}_\beta^c$ consisting of those $(z_\mathfrak{p})$ such that $|z_\mathfrak{p}|_\mathfrak{p} \leq 1$ for the finite places $\mathfrak{p} \mid (\beta)$.

Another big role in [MS14] is played by the natural extension of the beta-shift. Recall that the natural extension of a (non-invertible) dynamical system is an invertible dynamical system that contains the original dynamics as a subsystem and that is minimal with this property in a measure theoretical sense; it is unique up to metric isomorphism. If β is a Pisot number, then we obtain a geometric version of the natural extension of the beta-shift by suspending the Rauzy fractals; see Theorem 3.2. This natural extension domain characterises purely periodic beta-expansions [HI97, IR05, BS07] and forms (in the unit case) a Markov partition for the associated hyperbolic toral automorphism [Pra99], provided that it tiles the representation space periodically. The Pisot conjecture for beta-numeration can be stated as follows: the natural extension of the beta-shift is isomorphic to an automorphism of a compact group.

In the non-unit case, a third kind of compact sets, studied in [BSS+11] in the context of shift radix systems and similar to the intersective tiles in [ST13], turns out to be interesting. Integral beta-tiles are the Euclidean counterpart and can be seen as $\mathfrak{p}$-adic "slices" of Rauzy fractals. In Theorem 3.3, we provide some of their properties. In particular, we show that the boundary of these tiles has Lebesgue measure zero; this was conjectured in [BSS+11, Conjecture 7.1]. Furthermore they are intervals in the quadratic case and tile $\mathbb{R}$. This gives another proof of the well-known fact that the Pisot conjecture for two-letters substitutions is true.

One of the main results of [MS14] is the equivalence of the tiling property for all our collections of tiles. We extend the results from [IR06] to the beta-numeration case (where the associated substitution need not be irreducible or unit), with the restriction that the quotient mapping condition (QM) is needed for a periodic tiling with Rauzy fractals. Our series of equivalent tiling properties also contains that for the collection of integral beta-tiles. We complete then our Theorem 3.4 by proving the equivalence of these tiling properties with the weak finiteness property (W), and with a spectral criterion concerning the so-called boundary graph.

Finally, we make a thorough analysis of the properties of the number-theoretical function $\gamma(\beta)$ concerning the purely periodic beta-expansions. This function was defined in [Aki98] and is still not well understood; see [AFSS10], but note that the definition therein differs from ours for non-unit algebraic numbers. We improve in Theorem 3.5 some results of [ABBS08] and answer in Theorem 3.6 some of their posed questions for quadratic Pisot numbers.

Chapter 4 is based on [Min14], where we set up a geometrical theory for the dynamics of reducible Pisot substitutions. The main tools are the duals of higher dimensional extensions of substitutions, first introduced in [SAI01]. Generally we have that the number of letters (denoted by n) of the substitution is greater or equal than the degree d of the Pisot number, which is the dimension of the hyperbolic space $\mathbb{K}_\beta$ of the substitution. In particular, since we want to give a fractal geometric representation in the contracting space $\mathbb{K}_\beta^c$, we want to work with $(d-1)$-dimensional faces in $\mathbb{R}^n$, thus it turns out that the dual substitution $\mathbf{E}^*_{n-d+1}(\sigma)$, and its concrete geometric realization $\mathbf{E}^{d-1}(\sigma)$ defined as its conjugate by a sort of Poincaré duality map, will be suitable for this task.

Our main objects will be the Rauzy fractals defined as Hausdorff limits of renormalized patches of polygons generated by iterations of the dual substitution $\mathbf{E}^{d-1}(\sigma)$. Inspired by some ideas of [AFHI11] for the study of a free group automorphism associated with a complex Pisot root, we introduce some important geometrical conditions which are required in order to develop a tiling theory with these objects: *regularity* of the substitution guarantees that projecting $\mathbf{E}^{d-1}(\sigma)$-iterates of patches of $(d-1)$-dimensional faces behaves well without producing overlaps; the *geometric finiteness property* will ensure the covering property; finally an algebraic condition on the neutral polynomial of the substitution will imply nice topological properties, among which the measure disjointness in the set equations, for the Rauzy fractals. With these ingredients, iterating $\mathbf{E}^{d-1}(\sigma)$ on increasing patches of faces we succeed to produce geometrical representations for stepped surfaces whose projections onto the contracting space are polygonal tilings. These are the contents of Theorem 4.11. Under the same conditions we show in Theorem 4.20 that our Rauzy fractals form *aperiodic self-replicating tilings*, by just replacing the polygons in the polygonal tiling induced by a stepped surface. Furthermore, starting with patches of polygons $\mathcal{P}$ whose projections tile periodically $\mathbb{K}_\beta^c$ and operating with a Hausdorff limit process as described above, we get in Theorem 4.24 natural *periodic tilings* by Rauzy fractals $\mathcal{R}_\mathcal{P}$ whenever the boundaries of the approximations converge to the real boundary. We emphasise that explicit general constructions of periodic tilings were missing in the previous works on reducible substitutions.

Under a slight generalization of the strong coincidence condition it is possible to define a domain exchange transformation on our new Rauzy fractals, and we are interested in codings of orbits of points under this domain exchange. Furthermore, our new fractals turn out to be exactly those extended domains considered in [EI05, BBK06] obtained by taking into account the pieces prior to the first return of a domain exchange on the classical Rauzy fractal, with the advantage that they are generated explicitly in a systematic way by the dual substitution $\mathbf{E}^{d-1}(\sigma)$.

We present a new approach based on *broken lines*, trying to pursue the direction of turning a reducible substitution into an irreducible one. It is well known that the Rauzy fractal can be defined as the closure of the projections of vertices of a broken line which represents geometrically the fixed point of the substitution. By applying a code we can change combinatorially the broken line into another one where only some specific letters are used. This code is the combinatorial interpretation of some linear dependencies arising in the reducible case. Thus, applying this code turns in some sense the broken line into an irreducible one, where only the letters associated with the linearly independent basis vectors are used. Projecting the vertices of the new broken line we get a bigger domain, which turns out to be exactly one of our Rauzy fractals $\mathcal{R}_{\mathcal{P}}$ generated by the dual $\mathbf{E}^{d-1}(\sigma)$ and inducing a periodic tiling. This code explains combinatorially also the first return of the pieces. We will see in Theorem 4.39 that the symbolic dynamical system image of (X_σ, S) by this code is measurably conjugate to the domain exchange on $\mathcal{R}_{\mathcal{P}}$.

We apply our techniques to a family of reducible Pisot substitutions (including the Hokkaido substitution) satisfying the geometrical conditions required to get the tiling properties, and we show also some non-regular examples.

CHAPTER 1

Preliminaries

1.1. Substitutions

Let $\mathcal{A} = \{1, 2, \dots, n\}$ be a finite alphabet, and denote by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$. The set $\mathcal{A}^*$ endowed with the concatenation of words is a free monoid with the empty word ϵ as identity element. Given $w \in \mathcal{A}^*$ and $a \in \mathcal{A}$, let $|w|$ be the length of the finite word w, $|w|_a$ be the number of occurrences of a in w. We denote by $\mathcal{A}^\omega$ the set of *right-infinite words* and by ${}^\omega\mathcal{A}$ the set of *left-infinite words* over $\mathcal{A}$. The topology on $\mathcal{A}^\omega$ is the product topology of the discrete topology on $\mathcal{A}$. This implies that $\mathcal{A}^\omega$ is a compact Cantor set. A *bi-infinite word* over $\mathcal{A}$ is a two-sided sequence in $\mathcal{A}^\mathbb{Z}$. We can equip $\mathcal{A}^\mathbb{Z}$ with a topology in an analogous way as we did for $\mathcal{A}^\omega$. A right or bi-infinite word u is *purely periodic* if there exists $v \in \mathcal{A}^* \setminus \{\epsilon\}$ such that $u = v^\omega$. The *language* of an infinite or bi-infinite word u is the set of all its finite subwords. Recall that u is *uniformly recurrent* if every word occurring in u occurs in an infinite number of positions with bounded gaps.

A *substitution* is an endomorphism of the free monoid $\mathcal{A}^*$ with the condition that the image of each letter is non-empty and, for at least one letter $a \in \mathcal{A}$, $|\sigma^k(a)| \to \infty$. A substitution naturally extends to the set of infinite and bi-infinite sequences. A one-sided (two-sided) *periodic point* of σ is an infinite (bi-infinite) word u that satisfies $\sigma^k(u) = u$, for some $k > 0$. If $k = 1$, then u is called *fixed point* of σ.

We can naturally associate with a substitution σ an *incidence matrix* M_σ with entries $(M_\sigma)_{a,b} = |\sigma(b)|_a$, for all $a, b \in \mathcal{A}$. The map $\mathbf{l} : \mathcal{A}^* \to \mathbb{N}^n$, $w \mapsto (|w|_1, \dots, |w|_n)^t$ is called the *abelianisation map*. Obviously, we have $M_\sigma \circ \mathbf{l} = \mathbf{l} \circ \sigma$. A substitution is *primitive* if M_σ is a primitive matrix, i.e. $\exists\, k$ such that $M_\sigma^k > 0$. Every primitive substitution σ has at least one periodic point and without loss of generality we can assume that σ has at least one fixed point. Indeed, if k is the period length then we may just work with σ^k instead of σ. According to the Perron-Frobenius Theorem, if σ is primitive then M_σ has a simple positive eigenvalue, which we call the *Perron-Frobenius eigenvalue*, which is larger than the absolute value of all other eigenvalues. Furthermore, there exists an eigenvector with positive entries associated with the Perron-Frobenius eigenvalue.

The *prefix-suffix graph* associated with the substitution σ is the directed graph with set of vertices $\mathcal{A}$ and set of labelled edges $a \xrightarrow{(p,s)} b$ if there exist $p, s \in \mathcal{A}^*$ such that $\sigma(a) = pbs$. The *prefix* and *suffix graph* are those with labelled edges $a \xrightarrow{p} b$ and $a \xrightarrow{s} b$ respectively.

We are interested in the class of Pisot substitutions. We introduce now all the necessary definitions.

DEFINITION 1.1. An algebraic integer $\beta > 1$ is a *Pisot number* if all its algebraic conjugates β' other than β itself satisfy $|\beta'| < 1$.

DEFINITION 1.2. Let σ be a (primitive) substitution with β dominant eigenvalue of M_σ. We say that σ is

- *Pisot* if β is a Pisot number.
- *irreducible* if the characteristic polynomial of M_σ is irreducible over $\mathbb{Q}$, otherwise we call it *reducible*.
- *unit* if β is a unit, i.e. $N(\beta) = \pm 1$, otherwise we call it *non-unit*.

Given a Pisot substitution σ suppose that the characteristic polynomial of M_σ decomposes over $\mathbb{Q}$ into irreducible factors as

$$\det(xI - M_\sigma) = f(x)g_1(x)^{m_1} \cdots g_k(x)^{m_k},$$

where $f(x)$ is the minimal polynomial of degree d of the Pisot root β. We call $f(x)$ the *Pisot polynomial* and $g(x) := g_1(x)^{m_1} \cdots g_k(x)^{m_k}$ the *neutral polynomial*.

Each irreducible Pisot substitution is primitive (see e.g. [CS01b]). We introduce the following important combinatorial condition on substitutions introduced in [AI01].

DEFINITION 1.3. A substitution σ over the alphabet $\mathcal{A}$ satisfies the *strong coincidence condition* if for every pair $(b_1, b_2) \in \mathcal{A}^2$, there exists $k \in \mathbb{N}$ and $a \in \mathcal{A}$ such that $\sigma^k(b_1) = p_1 a s_1$ and $\sigma^k(b_2) = p_2 a s_2$ with $\mathbf{l}(p_1) = \mathbf{l}(p_2)$ or $\mathbf{l}(s_1) = \mathbf{l}(s_2)$.

Every Pisot substitution on two letters satisfies the strong coincidence condition (see [BD02]).

1.1.1. Substitution dynamical systems. For standard terminology and concepts about topological and measure-theoretical dynamical systems we refer to [Wal82, CFS82, EW11]. We recall some background notions of symbolic dynamical systems (for more details see [LM95]). The *two-sided shift* $S : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ is defined by $S(x_i)_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}$, and is a homeomorphism on $\mathcal{A}^\mathbb{Z}$. A *subshift*, or *shift space*, is a dynamical system (X, S) where $X \subseteq \mathcal{A}^\mathbb{Z}$ is a closed S-invariant set. Equivalently, there exists a set of forbidden words $\mathcal{F}$ such that X is the set of infinite sequences which do not contain any forbidden word in $\mathcal{F}$. A subshift is *of finite type* if the set of forbidden words $\mathcal{F}$ is finite. A subshift is *sofic* if its language is recognized by a deterministic finite automaton. Note that every subshift of finite type is sofic.

DEFINITION 1.4. Let σ be a primitive substitution. The *symbolic dynamical system generated by* σ is the shift space (X_σ, S) where

$$X_\sigma = \overline{\{S^k u : k \in \mathbb{Z}\}},$$

and $u \in \mathcal{A}^\mathbb{Z}$ is a fixed point of σ.

Observe that (X_σ, S) is made of all the two-sided sequences whose language coincides with the language of u, which does not depend on the choice of u by primitivity, since all σ-periodic words are uniformly recurrent and thus have the same language. We know that (X_σ, S) is *minimal* (every orbit is dense), *uniquely ergodic* (there is a unique ergodic shift-invariant Borel probability measure on X_σ) with *zero entropy* (the subword complexity of sequences in X_σ is linear). For more details see [Fog02, Que10].

We will need especially in Section 2.3.2 the following desubstitution theory [Mos92]. Every word in X_σ has a unique decomposition $w = S^k(\sigma(v))$, with $v \in X_\sigma$ and $0 \le k < |\sigma(v_0)|$. This means that any word in X_σ can be uniquely written in the form

$$w = \cdots \mid \underbrace{\cdots}_{\sigma(v_{-1})} \mid \underbrace{w_{-k} \cdots w_{-1}.w_0 \cdots w_l}_{\sigma(v_0)} \mid \underbrace{\cdots}_{\sigma(v_1)} \mid \underbrace{\cdots}_{\sigma(v_2)} \mid \cdots$$

with $\cdots v_{-1} v_0 v_1 \cdots \in X_\sigma$. Let $p = w_{-k} \cdots w_{-1}$ the prefix of $\sigma(v_0)$ of length k and let $s = w_1 \cdots w_l$ its suffix. The word w is completely defined by the word v and the decomposition of $\sigma(v_0)$ of the form pw_0s. Let $\mathcal{P}$ be the finite set of all such decompositions, i.e.,

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^* : \exists\, b \in \mathcal{A}, \sigma(b) = pas\}. \tag{1.1}$$

We can define a *desubstitution map* ϑ on X_σ (which sends w to v), and a partition map ρ from X_σ to $\mathcal{P}$, corresponding to the decomposition of $\sigma(v_0)$:

$$\vartheta : X_\sigma \to X_\sigma,\ w \mapsto v \quad \text{such that} \quad w = S^k\sigma(v) \ \text{ and } \ 0 \le k < |\sigma(v_0)|,$$
$$\rho : X_\sigma \to \mathcal{P},\ w \mapsto (p, w_0, s) \quad \text{such that} \quad \sigma(v_0) = pw_0s \ \text{ and } \ k = |p|.$$

Let $X^l_{\mathcal{P}}$ be the set of left-infinite sequences

$$(p_i, a_i, s_i)_{i \ge 0} = \cdots (p_1, a_1, s_1)(p_0, a_0, s_0) \in {}^\omega\mathcal{P}$$

such that $\sigma(a_{i+1}) = p_i a_i s_i$, for all $i \ge 0$. If we project each of the (p_i, a_i, s_i) of an element of $X^l_{\mathcal{P}}$ on the first component we obtain the labels of a left-infinite walk in the prefix graph of the substitution σ. The subshift $X^l_{\mathcal{P}}$ is sofic. The *prefix-suffix development* is the map $\psi_{\mathcal{P}} : X_\sigma \to X^l_{\mathcal{P}}$ defined by $\psi_{\mathcal{P}}(w) = (\rho(\vartheta^i(w)))_{i \ge 0} = (p_i, a_i, s_i)_{i \ge 0}$. If an infinite number of prefixes and suffixes are non-empty then we have the combinatorial expansion

$$w = \lim_{k \to \infty} \sigma^k(p_k) \cdots \sigma(p_1) p_0 . w_0 s_0 \sigma(s_1) \cdots \sigma^k(s_k), \tag{1.2}$$

where the triples (p_i, a_i, s_i) play the role of digits. It is shown in [CS01a] that the map $\psi_{\mathcal{P}}$ is continuous and onto $X^l_{\mathcal{P}}$, and it is one-to-one except on the orbit of periodic points of σ, where it is k-to-one with $k > 1$.

1.2. Numeration

Fractals and numeration systems are closely related. Bear in mind that the *Cantor set* is the set of elements $\sum_{i \ge 1} d_i 3^{-i}$ with $d_i \in \{0, 2\}$. A more sophisticated example is given by *Knuth's numeration system* [Knu98]. Every element of the ring of integers $\mathbb{Z}[i]$ of the field of Gaussian numbers $\mathbb{Q}(i)$ can be uniquely represented as $\sum_{k=0}^m d_k(-1+i)^k$, with $d_k \in \{0, 1\}$. Precisely we say that $(-1+i, \{0, 1\})$ is a *canonical number system* for $\mathbb{Z}[i]$. The set of "fractional parts" of this numeration system

$$\mathcal{T} = \left\{ \sum_{k \ge 1} d_k(-1+i)^{-k} \in \mathbb{C} : d_k \in \{0, 1\} \right\}$$

is a well-known fractal, called the *twin dragon*. It has nice properties like compactness, it is the closure of its interior, its boundary is a fractal set with

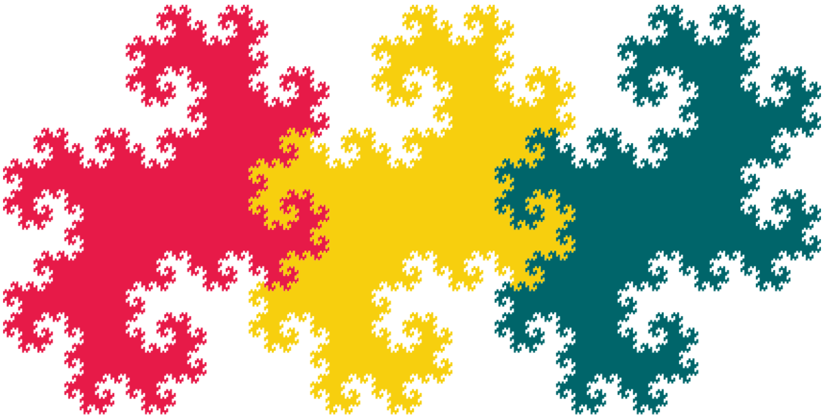


FIGURE 1.1. Periodic tiling induced by the twin dragon.

measure zero, and it is a self-similar set since directly from the definition we can see that it satisfies the set equation

$$\mathcal{T} = b^{-1}\mathcal{T} \cup b^{-1}(\mathcal{T}+1) \qquad (b=-1+i).$$

Furthermore it induces a lattice tiling of $\mathbb{C}$ in the sense that

$$\bigcup_{z\in\mathbb{Z}[i]} \mathcal{T}+z = \mathbb{C},$$

and $(\mathcal{T}+z_1)\cap(\mathcal{T}+z_2)$ has zero Lebesgue measure if $z_1 \neq z_2$.

We will introduce in the next sections two numeration systems: *Dumont-Thomas numeration* and *beta-numeration*. We will see in the sequel that we can associate with these numeration systems a geometrical representation, more precisely some fractal tiles, which will have similar properties as the twin dragon.

1.2.1. Dumont-Thomas numeration. Dumont and Thomas [DT89] studied numeration systems associated with a primitive substitution σ.

Every finite prefix of a one-sided fixed point u of σ can be uniquely expanded as

$$\sigma^k(p_k)\sigma^{k-1}(p_{k-1})\cdots\sigma(p_1)p_0, \tag{1.3}$$

where $(p_i)_{i=0}^k$, $p_k \neq \epsilon$, is a walk in the prefix graph of σ starting from u_0, that is $\sigma(u_0) = p_k a_k s_k$, $\sigma(a_i) = p_{i-1}a_{i-1}s_{i-1}$ for all $1 \le i \le k$. Thus we recover numeration defined on $\mathbb{N}$ by expanding the length N of a finite prefix of u as $N = |\sigma^k(p_k)| + \cdots + |p_0|$.

This notion of numeration allows to expand real numbers with respect to a real base $\beta > 1$, which is the Perron-Frobenius eigenvalue of the substitution. Dumont-Thomas expansions depend on the prefix graph of the substitution and on the left eigenvector $\mathbf{v}_\beta$ associated with β. The digit set for the expansions is $\mathcal{D} = \{v_p : (p,a,s) \in \mathcal{P}\}$, where $\mathcal{P}$ is defined in (1.1) and v_p denotes $\langle \mathbf{l}(p), \mathbf{v}_\beta\rangle$.

A sequence $(v_{p_i})_{i\ge1} \in \mathcal{D}^\omega$ is called (σ,a)-*admissible* if there exists a walk in the prefix graph labelled by $(p_i)_{i\ge1}$ starting from a with infinitely many non-empty suffixes.

Proposition 1.5 ([DT89]). *Let σ be a primitive substitution on the alphabet $\mathcal{A}$ and fix $a \in \mathcal{A}$. For every $x \in [0, v_a)$, there exists a unique (σ, a)-admissible sequence $(v_{p_i})_{i\geq 1} \in \mathcal{D}^\omega$ such that*

$$x = \sum_{i\geq 1} v_{p_i}\beta^{-i}. \tag{1.4}$$

We will call an expansion of this form a (σ, a)-*expansion* and we will denote its sequence of digits by $(x)_{\sigma,a}$.

Set $X = \bigcup_{a\in\mathcal{A}} \big([0, v_a) \times \{a\}\big)$ and define the map

$$T_\sigma : X \to X, \quad (y, b) \mapsto \big(\beta y - v_p, a\big),$$

where $a \in \mathcal{A}$ and $p \in \mathcal{A}^*$ are uniquely determined by $\sigma(b) = pas$ and $\beta y - v_p \in [0, v_a)$. Given any $(y, b) \in X$ we get its (σ, b)-expansion by computing its T_σ-orbit. Dumont-Thomas numeration is an example of *fibred numeration system* (for more details see [BBLT06, Section 4.1]).

Notice that T_σ is not injective and the pre-image has the form

$$T_\sigma^{-1}(x, a) = \bigcup_{b\xrightarrow{p}a} \{(\beta^{-1}(x + v_p), b)\}, \quad \text{for } (x, a) \in X. \tag{1.5}$$

It is easy to see from this identity that for $(x, a) \in X$ we have

$$\beta^m T_\sigma^{-m}(x, a) = x + \beta^m T_\sigma^{-m}(0, a), \quad \forall m \in \mathbb{N}, \tag{1.6}$$

where $x + (z, a) = (x + z, a)$ is used.

We will be interested in "integers" and "fractional parts" obtained from the Dumont-Thomas numeration system generated by the substitution σ, i.e., all those $x \in \mathbb{R}^+$ such that only non-negative (respectively negative) powers of β occur in their (σ, a)-expansions, for some $a \in \mathcal{A}$.

We introduce topological limits (see e.g. [Kur66, §29]).

Definition 1.6. Let (A_n) be a collection of sets in a topological space. A point z belongs to the *lower limit* $\underline{\mathrm{Lim}}\, A_n$ if every neighbourhood of z intersects all the A_n for n sufficiently large. A point z belongs to the *upper limit* $\overline{\mathrm{Lim}}\, A_n$ if every neighbourhood of z intersects A_n for infinitely many values of n. If $A = \underline{\mathrm{Lim}}\, A_n = \overline{\mathrm{Lim}}\, A_n$, we call $A = \mathrm{Lim}\, A_n$ the *topological limit* of (A_n).

Let $\mathbb{Z}_{\sigma,a}^{(k)} = \beta^k T_\sigma^{-k}(0, a) \subset \mathbb{R}$ be the set of real numbers corresponding to all finite walks of length k in the prefix graph ending at state a, i.e. the sums $\sum_{i=0}^{k-1} v_{p_i}\beta^i$, where $a_k \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_1} a_1 \xrightarrow{p_0} a$. To such an element we associate the left-sequence of digits $v_{p_{k-1}} \cdots v_{p_1} v_{p_0}. \in {}^*\mathcal{D}$. Observe that these sets are not nested (see Example 1.12).

Definition 1.7. The set of (σ, a)-*integers* is the topological limit

$$\mathbb{Z}_{\sigma,a} = \mathrm{Lim}_{k\to\infty}\, \mathbb{Z}_{\sigma,a}^{(k)}. \tag{1.7}$$

We call the union $\bigcup_{a\in\mathcal{A}} \mathbb{Z}_{\sigma,a}$ the σ-*integers* and denote it by $\mathbb{Z}_\sigma$.

The topological limit in the definition exists since for every interval $[0, \ell] \subset \mathbb{R}^+$ there exists $k_0 \in \mathbb{N}$ such that $\mathrm{Lim}_{k\to\infty} \mathbb{Z}_{\sigma,a}^{(k)} \cap [0, \ell] = \mathbb{Z}_{\sigma,a}^{(k)} \cap [0, \ell]$, for each $k \geq k_0$.

Notice that the set of σ-integers is discrete and closed. The set $\mathbb{Z}_{\sigma,a}$ is the set of those finite sums $\sum_{i=0}^{k-1} v_{p_i}\beta^i \in \mathbb{Z}_{\sigma,a}^{(k)}$, $k \in \mathbb{N}$, whose associated sequence of digits can be left-padded by zeros. In particular, $\mathbb{Z}_{\sigma,a} \subsetneq \bigcup_{k\geq 0} \mathbb{Z}_{\sigma,a}^{(k)}$, that is, not every truncation is a (σ, a)-integer.

Definition 1.8. Let $\mathbf{v}_\beta = (v_1, \ldots, v_n)$ be a left eigenvector of M_σ associated with β, and assume that $\mathbf{v}_\beta$ is scaled in a way such that each $v_i \in q^{-1}\mathbb{Z}[\beta]$ for some $q \in \mathbb{Z}$. We denote the $\mathbb{Z}$-module generated by the v_i by

$$V_\mathbb{Z} = \langle v_1, \ldots, v_n \rangle_\mathbb{Z} \subseteq q^{-1}\mathbb{Z}[\beta]. \tag{1.8}$$

Note that the σ-integers form a subset of $V_\mathbb{Z}$.

Lemma 1.9. *$V_\mathbb{Z}$ is a free $\mathbb{Z}$-module of rank $d = \deg(\beta)$.*

Proof. As $\mathbf{v}_\beta$ is an eigenvector, we get that $\beta V_\mathbb{Z} \subset V_\mathbb{Z}$. Moreover, $\mathbf{v}_\beta \neq \mathbf{0}$ which implies that $V_\mathbb{Z} \neq \{0\}$. Thus, since β is irrational of degree d, the elements $v_j, \beta v_j, \ldots, \beta^{d-1} v_j \in V_\mathbb{Z}$ are linearly independent over $\mathbb{Q}$. Therefore $\langle v_j, \beta v_j, \ldots, \beta^{d-1} v_j \rangle_\mathbb{Z} \subset V_\mathbb{Z} \subseteq q^{-1}\mathbb{Z}[\beta]$ and, hence, V has rank d. □

Since $V_\mathbb{Z}$ is an Abelian group and $\beta V_\mathbb{Z} \subset V_\mathbb{Z}$ we have that $V_\mathbb{Z}$ is a finitely generated $\mathbb{Z}[\beta^{-1}]$-module and $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ is a (fractional) ideal of the ring $\mathbb{Z}[\beta^{-1}]$.

Definition 1.10. The set of (σ, a)-*fractional parts* is defined as

$$\mathrm{Frac}(\sigma, a) = V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, v_a), \tag{1.9}$$

and $\mathrm{Frac}(\sigma) = \bigcup_{a\in\mathcal{A}} \mathrm{Frac}(\sigma, a) = V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, \max_{a\in\mathcal{A}} v_a)$, will be called the set of σ-*fractional parts*.

An element $x \in \mathrm{Frac}(\sigma, a)$ has a (σ, a)-expansion $(x)_{\sigma,a} = .v_{p_{-1}} v_{p_{-2}} \cdots$, where $(p_{-k})_{k\geq 1}$ is the label of an infinite walk in the prefix graph starting at state a.

Lemma 1.11. *T_σ maps Frac(σ) onto Frac(σ).*

Proof. Given $(x, a) \in \mathrm{Frac}(\sigma, a)$, $T_\sigma(y, b) = (x, a)$ for all (y, b) such that $y = \beta^{-1}(x + v_p)$ and $\sigma(b) = pas$. Notice that there exists at least one (y, b) of this form since the prefix graph is strongly connected by the primitivity of σ. It is clear that $y \in V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$. Furthermore, if $(x)_{\sigma,a} = .v_{p_1} v_{p_2} \cdots$, then $(y)_{\sigma,b} = .v_p v_{p_1} v_{p_2} \cdots$ which implies that $y \in [0, v_b)$. □

Example 1.12. Let σ be the substitution $\sigma(1) = 121$, $\sigma(2) = 11$. We have

$$M_\sigma = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \det(xI - M_\sigma) = x^2 - 2x - 2.$$

This substitution is an irreducible non-unit Pisot substitution with associated Pisot root $\beta = 1 + \sqrt{3}$. A left eigenvector associated with β for M_σ is $\mathbf{v}_\beta = (\frac{\beta}{2}, 1)$. From the prefix graph of the substitution depicted in Figure 1.2 we can see that the set of digits is $\mathcal{D} = \{0, v_1, v_{12}\}$.

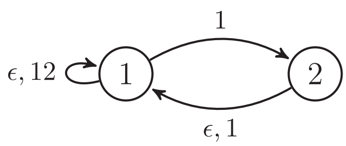


Figure 1.2. The prefix graph of the substitution σ.

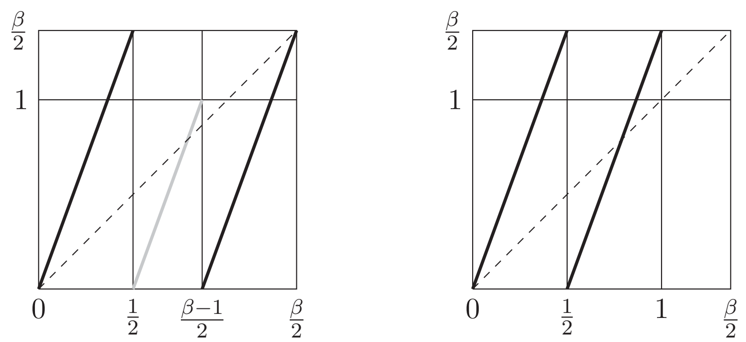


FIGURE 1.3. The map T_σ.

In Figure 1.3 we illustrate the map T_σ and its combinatorial structure. Given a point $(x, a) \in X$, if $a = 1$ then the function depicted in the left square is used to compute the image by T_σ, if $a = 2$ we use the one in the right square. Furthermore, $T_\sigma(x, a) = (y, 1)$ if we encounter a black linear piece of the map, and equals $(y, 2)$ if it is light gray. For example, given $(x, 1) \in [1, \frac{\beta-1}{2}) \times \{1\}$, which is in the left square, after one iteration of T_σ it will jump to the right square. We compute as an example the orbit of $(\frac{1}{4}, 1)$:

$$(\tfrac{1}{4}, 1) \xrightarrow{T_\sigma} (\tfrac{\beta}{4}, 1) \xrightarrow{T_\sigma} (\tfrac{1}{2}, 2) \xrightarrow{T_\sigma} (0, 1) \circlearrowright T_\sigma$$

Thus we have $(\frac{1}{4})_{\sigma,1} = .0v_1v_1$. Observe that $(\frac{\beta^3}{4}, 1) = (\frac{3\beta}{2} + 1, 1) \in \mathbb{Z}^{(k)}_{\sigma,1}$ for all $k \geq 2$, thus $\frac{3\beta}{2} + 1$ is a $(\sigma, 1)$-integer. On the other hand $(\frac{3\beta}{2} + 1, 2) \in \mathbb{Z}^{(2)}_{\sigma,2}$, with associated walk $2 \xrightarrow{\delta(1)} 1 \xrightarrow{\delta(1)} 2$, but $(\frac{3\beta}{2} + 1, 2) \notin \mathbb{Z}^{(3)}_{\sigma,2}$, and this is due to the fact that we cannot left-pad its expansion by 0's, i.e., we can extend the walk in the automaton on the left only by adding a digit v_1. As another example we have $v_1v_{12}. \in \mathbb{Z}^{(2)}_{\sigma,1}$, with associated walk $2 \xrightarrow{v_1} 1 \xrightarrow{v_{12}} 1$, but it cannot be extended to any infinite walk. In this sense, it remains an approximation. We list some other expansions:

$$(\tfrac{\beta-1}{3})_{\sigma,2} = .(v_1 0)^\omega \qquad (\beta - 2)_{\sigma,1} = .v_1v_1(0\,v_{12})^\omega, \qquad (\tfrac{\beta-1}{4})_{\sigma,1} = .0\,v_{12}\,v_{12}$$

1.2.2. Beta-numeration. Let $\beta > 1$ be a real number. The map

$$T_\beta : \ [0, 1) \to [0, 1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor, \tag{1.10}$$

is the classical *greedy β-transformation*. Each $x \in [0, 1)$ has a unique *(greedy) β-expansion*

$$x = \sum_{k=1}^{\infty} d_k \beta^{-k}, \quad \text{with} \quad d_k = \lfloor \beta\, T_\beta^{k-1}(x) \rfloor;$$

the digits d_k are in $\mathcal{D} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$. The β-expansions are called greedy since, for each $k \geq 1$, the chosen digit d_k is always the greatest possible of $\mathcal{D}$ such that $\sum_{i=1}^{k} d_i \beta^{-i} \leq x$.

Define $(\cdot)_\beta : [0, 1) \to \mathcal{D}^{\mathbb{N}}$, $x \mapsto d_1 d_2 \cdots$ where the d_k are the digits of the β-expansion of x. The set of admissible sequences was characterised first by Parry [Par60] and depends only on the expansion of 1. The β-expansion of 1 can be defined considering the iterations $T_\beta^k(1^-) = \lim_{x\to 1^-} T_\beta^k(x)$. We denote its sequence of digits by $(1^-)_\beta$. Then the set of greedy β-expansions of numbers of $[0,1)$ is exactly the set of sequences $(d_k)_{k\geq 1} <_{\text{lex}} (1^-)_\beta$ in lexicographical order. This is a shift-invariant subset of $\mathcal{D}^{\mathbb{N}}$. The action of the shift on the closure of this set, which consists in the sequences $(d_k)_{k\geq 1} \leq_{\text{lex}} (1^-)_\beta$, forms a subshift called *β-shift*.

Definition 1.13. Numbers β such that $(1^-)_\beta$ is ultimately periodic are called *Parry numbers* and those such that $(1^-)_\beta$ is purely periodic are called *simple Parry numbers*.

The dynamical system $([0,1), T_\beta)$ is conjugate to the β-shift, which is sofic if and only if β is a Parry number, or of finite type if and only if β is a simple Parry number [IT74, BM86, Bla89]. Furthermore the map T_β is ergodic with an absolute continuous invariant measure, in particular weak-mixing, with a unique measure of maximal entropy. It can be shown that $([0,1), T_\beta)$ is weakly Bernoulli and its natural extension is a Bernoulli automorphism (for more details see [DK02]).

Pisot numbers are Parry numbers since every element of $\mathbb{Q}(\beta)\cap[0,1]$ has an eventually periodic β-expansion [Ber77, Sch80].

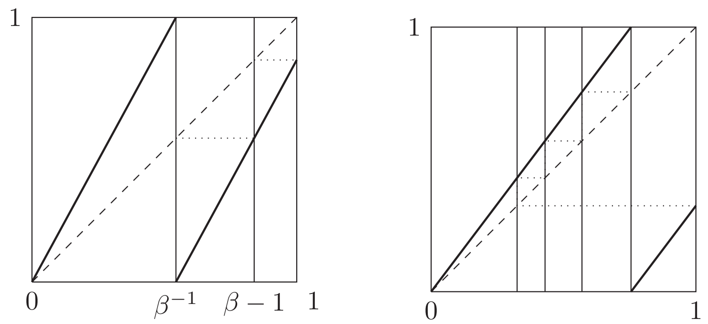


Figure 1.4. T_β for $\beta^3 = \beta^2+\beta+1$ (left) and for $\beta^3 = \beta + 1$ (right).

For β Pisot, we can associate with $([0,1), T_\beta)$ a *β-substitution* σ_β defined according to the two cases when β is a simple Parry number, that is $(1^-)_\beta = .(d_1\cdots d_{n-1}(d_n-1))^\omega$, or β is a non-simple Parry number, that is $(1^-)_\beta = .d_1\cdots d_\ell(d_{\ell+1}\cdots d_n)^\omega$.

$$\begin{aligned} \sigma_\beta(1) &= 1^{d_1}2 \\ \sigma_\beta(2) &= 1^{d_2}3 \\ &\vdots \\ \sigma_\beta(n) &= 1^{d_n} \end{aligned} \qquad\qquad \begin{aligned} \sigma_\beta(1) &= 1^{d_1}2 \\ \sigma_\beta(2) &= 1^{d_2}3 \\ &\vdots \\ \sigma_\beta(n-1) &= 1^{d_{n-1}}n \\ \sigma_\beta(n) &= 1^{d_n}(\ell+1) \end{aligned}$$

Proposition 1.14. *Let $(x-\beta)(x^{n-1}+v_2x^{n-2}+\cdots+v_n)$ be the characteristic polynomial of the incidence matrix M_{σ_β} of σ_β. Then $v_k = T_\beta^{k-1}(1^-)\in\mathbb{Z}[\beta]$ and $\mathbf{v}_\beta = (1, v_2, \ldots, v_n)$ is a left eigenvector of M_{σ_β} associated with β.*

Proof. We show it for simplicity only for simple Parry numbers. The characteristic polynomial of M_{σ_β} is $x^n - d_1x^{n-1} - \cdots - d_n$, thus we have the relations $d_k = \beta v_k - v_{k+1}$, for $1 \le k \le n-1$. Since $v_1 = 1$ and $(1^-)_\beta = .(d_1 \cdots d_{n-1}(d_n - 1))^\omega$ we deduce the equalities $v_k = T_\beta^{k-1}(1^-)$. A simple calculation shows that $(1, v_2, \ldots, v_n)$ is a left eigenvector of M_{σ_β}. □

Remark 1.15. Recall the definition (1.8) of $V_\mathbb{Z}$. By Proposition 1.14 we can choose

$$\mathbf{v}_\beta = (1, T_\beta(1^-), \ldots, T_\beta^{n-1}(1^-)).$$

Thus $V_\mathbb{Z} = \mathbb{Z}[\beta]$ and $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ is simply $\mathbb{Z}[\beta^{-1}]$. We will use this especially in Chapter 3.

Remark 1.16. Dumont-Thomas numeration associated with β-substitutions coincides with beta-numeration since the prefixes occurring in the prefix graph of σ_β are only strings made of 1's and $v_1 = 1$.

We will consider in Chapter 3 for β Pisot the sets of points of discontinuity

$$\widehat{V} = \big\{T_\beta^k(1^-) : k \ge 0\big\}, \qquad V = \big(\widehat{V} \cup \{0\}\big) \setminus \{1\}, \tag{1.11}$$

which are finite because each Pisot number is a Parry number. For $x \in [0,1)$, let

$$\widehat{x} = \min\big\{y \in \widehat{V} : y > x\big\}.$$

Thus, for $v \in V$, $\widehat{v}$ is the successor of v in $V \cup \{1\}$, and $\widehat{V} = \{\widehat{v} : v \in V\}$. Let

$$L = \big\langle \widehat{V} - \widehat{V} \big\rangle_\mathbb{Z} \subseteq \mathbb{Z}[\beta] \tag{1.12}$$

be the $\mathbb{Z}$-module generated by the differences of elements in $\widehat{V}$. The following condition

$$\operatorname{rank}(L) = \deg(\beta) - 1 \tag{QM}$$

is an analogue of the *quotient mapping condition* defined in [ST09]; see also the definition of the anti-diagonal torus in [BBK06, Section 8]. It is related to a periodic collection of tiles and will be important in Chapter 3.

1.3. Representation spaces

In all the following, let β be a Pisot number. Let $K = \mathbb{Q}(\beta)$, $\mathcal{O}$ its ring of integers. A *place* (or *prime*) $\mathfrak{p}$ is a class of equivalent valuations of K. For each (finite or infinite) prime $\mathfrak{p}$ of K, we choose an absolute value $|\cdot|_\mathfrak{p}$ and write $K_\mathfrak{p}$ for the completion of K with respect to $|\cdot|_\mathfrak{p}$. In all what follows, the absolute value $|\cdot|_\mathfrak{p}$ is chosen in the following way. Let $\xi \in K$ be given. If $\mathfrak{p} \mid \infty$, denote by $\xi^{(\mathfrak{p})}$ the associated Galois conjugate of ξ. If $\mathfrak{p}$ is real, we set $|\xi|_\mathfrak{p} = |\xi^{(\mathfrak{p})}|$, and if $\mathfrak{p}$ is complex, we set $|\xi|_\mathfrak{p} = |\xi^{(\mathfrak{p})}|^2$. Finally, if $\mathfrak{p}$ is finite, we put $|\xi|_\mathfrak{p} = \mathfrak{N}(\mathfrak{p})^{-v_\mathfrak{p}(\xi)}$, where $\mathfrak{N}(\cdot)$ is the norm of a (fractional) ideal and $v_\mathfrak{p}(\xi)$ denotes the exponent of $\mathfrak{p}$ in the prime ideal decomposition of the principal ideal (ξ). For more details we refer to [Neu99].

Set $S = \{\mathfrak{p} : \mathfrak{p} \mid \infty \text{ or } \mathfrak{p} \mid (\beta)\}$ and define the *representation space*

$$\mathbb{K}_\beta = \prod_{\mathfrak{p} \in S} K_\mathfrak{p} = \mathbb{K}_\infty \times \mathbb{K}_\mathrm{f}, \quad \text{with} \quad \mathbb{K}_\infty = \prod_{\mathfrak{p} \mid \infty} K_\mathfrak{p}, \quad \mathbb{K}_\mathrm{f} = \prod_{\mathfrak{p} \mid (\beta)} K_\mathfrak{p}.$$

If β has r real and s pairs of complex Galois conjugates, then $\mathbb{K}_\infty = \mathbb{R}^r \times \mathbb{C}^s$. The space $\mathbb{K}_\mathrm{f}$ is the product of $\mathfrak{p}$-adic spaces, which are finite extensions of $\mathbb{Q}_p$, for $\mathfrak{p} \mid (p)$. We equip $\mathbb{K}_\beta$ with the product metric of the metrics defined by the absolute values $|\cdot|_\mathfrak{p}$ and the product measure μ of the Haar measures $\mu_\mathfrak{p}$ on $K_\mathfrak{p}$, $\mathfrak{p} \in S$. We know that for every measurable subset M of $K_\mathfrak{p}$ and for every $x \in K_\mathfrak{p}$,

$$\mu_\mathfrak{p}(x \cdot M) = |x|_\mathfrak{p}\, \mu_\mathfrak{p}(M)$$

(see for instance [Ser79, Chapter II]). The elements of $\mathbb{Q}(\beta)$ are naturally represented in $\mathbb{K}_\beta$ by the diagonal embedding

$$\delta : \mathbb{Q}(\beta) \to \mathbb{K}_\beta, \quad \xi \mapsto \prod_{\mathfrak{p}\in S} \xi.$$

Let $\mathfrak{p}_1$ be the infinite place corresponding to the identical Galois automorphism, that is, $|\beta|_{\mathfrak{p}_1} = \beta$. We have an expanding-contracting decomposition given by

$$\mathbb{K}_\beta = \mathbb{K}_\beta^e \times \mathbb{K}_\beta^c = K_{\mathfrak{p}_1} \times \prod_{\mathfrak{p}\in S\setminus\{\mathfrak{p}_1\}} K_\mathfrak{p},$$

where $\mathbb{K}_\beta^e \cong \mathbb{R}$ is the *expanding space*, since $|\beta|_{\mathfrak{p}_1} > 1$, and $\mathbb{K}_\beta^c$ is the *contracting space*, since $|\beta|_\mathfrak{p} < 1$ for all $\mathfrak{p} \in S \setminus \{\mathfrak{p}_1\}$. We will also use the *stripe spaces*

$$Z = \mathbb{K}_\infty \times \overline{\delta_\mathrm{f}(\mathbb{Z}[\beta])} \quad \text{and} \quad Z^c = \mathbb{K}_\infty^c \times \overline{\delta_\mathrm{f}(\mathbb{Z}[\beta])},$$

where $\mathbb{K}_\infty^c$ denotes the product of $K_\mathfrak{p}$ for $\mathfrak{p} \mid \infty$, $\mathfrak{p} \neq \mathfrak{p}_1$.

Let π_1, $\pi_{S\setminus\{\mathfrak{p}_1\}}$ and π_∞^c be the canonical projections from $\mathbb{K}_\beta$ to $K_{\mathfrak{p}_1}$, $\mathbb{K}_\beta^c$, and $\mathbb{K}_\infty^c$ respectively (not to be confused with any of those in Section 1.3.2). The diagonal embeddings δ_e, δ_c, δ_∞, δ_∞^c, δ_f and the Haar measures μ_c, μ_∞^c are defined accordingly.

The action of $\mathbb{Q}(\beta)$ on any of the representation spaces we have seen up to now will be denoted by $\xi \cdot (z_\mathfrak{p}) = (\xi z_\mathfrak{p})$, for $\xi \in \mathbb{Q}(\beta)$. We will sometimes omit the dot in case it is clear from the context that we refer to this action.

Remark 1.17. If β is a unit there is no $\mathfrak{p} \mid (\beta)$, thus $\mathbb{K}_\beta = \mathbb{K}_\infty$.

1.3.1. Lattices and Delone sets. This section is about lattices and Delone sets which will form suitable translation sets for Rauzy fractals throughout this thesis.

We introduce a slightly more flexible version of the adèle ring. For a precise treatment we refer to [Cas67, Wei95, RV99]. Let K be a number field, $P(K)$ be the set of places of K and $P_\infty(K)$ be the set of infinite places.

Definition 1.18. Let S be such that $P_\infty(K) \subseteq S \subseteq P(K)$. The *ring of S-integers* is

$$\mathcal{O}_S = \{x \in K : |x|_\mathfrak{p} \le 1 \text{ for all } \mathfrak{p} \notin S\}.$$

For each finite set P of places $P_\infty(K) \subseteq P \subseteq S$ define the locally compact ring

$$\mathbb{A}_K(P) = \prod_{\mathfrak{p}\in P} K_\mathfrak{p} \times \prod_{\mathfrak{p}\in S\setminus P} \mathcal{O}_\mathfrak{p}$$

and define the *S-adèle ring* of K to be the ring

$$\mathbb{A}_{K,S} = \bigcup_{\substack{P \text{ finite} \\ P_\infty(K)\subseteq P\subseteq S}} \mathbb{A}_K(P) = \Big\{(x_\mathfrak{p})_{\mathfrak{p}\in S} \in \prod_{\mathfrak{p}} K_\mathfrak{p} : |x_\mathfrak{p}|_\mathfrak{p} \le 1 \text{ for almost all } \mathfrak{p} \in S\Big\},$$

with the topology defined by requiring that each $\mathbb{A}_K(P)$ is an open subring. We also write $\mathbb{A}_K = \mathbb{A}_{K,P(K)}$ for the *adèle ring* of K.

Notice that if $S = P(K)$, then $\mathcal{O}_S = K$. Denote again by δ the diagonal embedding of $\mathcal{O}_S$ into $\mathbb{A}_{K,S}$.

Lemma 1.19 (Approximation theorem, see e.g. [Wei95]). *For any number field K and set S with $P_\infty(K) \subseteq S \subseteq P(K)$,*

$$\mathbb{A}_{K,S} = \delta(\mathcal{O}_S) + \mathbb{A}_K(P_\infty(K)).$$

Moreover $\delta(\mathcal{O}_S)$ is discrete and co-compact in $\mathbb{A}_{K,S}$.

We go back now to our settings, where $K = \mathbb{Q}(\beta)$, $S = \{\mathfrak{p} : \mathfrak{p} \mid \infty \text{ or } \mathfrak{p} \mid (\beta)\}$.

Lemma 1.20. *The set $\delta(\mathcal{O}_S)$ is a lattice in $\mathbb{K}_\beta$.*

Proof. We have that $\mathbb{A}_{K,S} = \mathbb{K}_\beta$. Then $\delta(\mathcal{O}_S)$ is a lattice in $\mathbb{K}_\beta$ by the approximation theorem for number fields. □

Recall from (1.8) and Lemma 1.9 that $V_\mathbb{Z}$ is the $\mathbb{Z}$-module of rank $d = \deg(\beta)$ generated by the components of a left eigenvector $\mathbf{v}_\beta$ of M_σ associated with the Pisot number β

$$V_\mathbb{Z} = \langle v_1, \dots, v_n \rangle_\mathbb{Z},$$

and $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ is a (fractional) ideal of the ring $\mathbb{Z}[\beta^{-1}]$.

Lemma 1.21. *The following assertions hold:*

(1) $\mathcal{O}_S = \mathcal{O}[\beta^{-1}]$.
(2) $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ is a subgroup of finite index of $\mathcal{O}_S$.

Proof. Since $\beta^{-1} \in \mathcal{O}_S$ and $\mathcal{O} \subseteq \mathcal{O}_S$ the inclusion $\mathcal{O}_S \supseteq \mathcal{O}[\beta^{-1}]$ follows. To prove the reverse inclusion, choose $x \in \mathcal{O}_S$ and let $\mathfrak{p} \mid (\beta)$. Then there exists $k \in \mathbb{N}$ such that $|\beta^k x|_\mathfrak{p} \leq 1$. Since S is a finite set of primes, setting $h = \max\{k \in \mathbb{N} : |\beta^k x|_\mathfrak{p} \leq 1,\ \text{for } \mathfrak{p} \mid (\beta)\}$ we get $\beta^h x \in \mathcal{O}$, and, hence, $\mathcal{O}_S \subseteq \mathcal{O}[\beta^{-1}]$.

$V_\mathbb{Z}$ is a subgroup of finite index of $q^{-1}\mathcal{O}$, for some $q \in \mathbb{Z}$, which implies that there exists $m \in \mathbb{N}$ such that $mq^{-1}\mathcal{O} \subseteq V_\mathbb{Z}$. Thus $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}] \subseteq q^{-1}\mathcal{O}[\beta^{-1}] \subseteq \frac{1}{m} V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ and it suffices to show that $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ is a subgroup of finite index of $\frac{1}{m} V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$. Suppose on the contrary that $mV_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ is a subgroup of $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$ of infinite index, in particular $|V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]/mV_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]| > m^d$. Let $x_1, \dots, x_{m^d+1}$ be $m^d + 1$ pairwise different representatives of $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]/mV_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$. Since $x_1, \dots, x_{m^d+1} \in V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$, there exists $l \in \mathbb{N}$ such that $x_1, \dots, x_{m^d+1} \in V_\mathbb{Z}\langle 1, \beta^{-1}, \dots, \beta^{-l}\rangle_\mathbb{Z}$. As $V_\mathbb{Z}\langle 1, \beta^{-1}, \dots, \beta^{-l}\rangle_\mathbb{Z}$ is a $\mathbb{Z}$-module of rank at most d, $V_\mathbb{Z}\langle 1, \beta^{-1}, \dots, \beta^{-l}\rangle_\mathbb{Z}/mV_\mathbb{Z}\langle 1, \beta^{-1}, \dots, \beta^{-l}\rangle_\mathbb{Z}$ has index at most m^d, which implies that there exist $i, j \in \{1, \dots, m^d + 1\}$ such that $x_i \equiv x_j \bmod mV_\mathbb{Z}\langle 1, \beta^{-1}, \dots, \beta^{-l}\rangle_\mathbb{Z}$, contradicting $x_i \not\equiv x_j \bmod mV_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$. □

Definition 1.22. A subset X of a metric space is *uniformly discrete* if there is a radius $r > 0$ such that each ball of radius r contains at most one point of X. A metric space X is *relatively dense* if there is a radius $R > 0$ such that each ball of radius R contains at least one point of X. A *Delone set* is a uniformly discrete and relatively dense set.

For example, lattices are Delone sets with additional group structure.

Lemma 1.23. *The set $\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$ is a lattice in $\mathbb{K}_\beta$. Furthermore, each set $\delta_c(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap X)$, where $X \subset \mathbb{R}$ is bounded and has non-empty interior, is a Delone set in $\mathbb{K}_\beta^c$.*

Proof. The first statement is a direct consequence of Lemma 1.20 and Lemma 1.21. Note that $\delta_c(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap X)$ is a model set (see Section 2.2.4 for a definition). We refer to [KS12, Lemma 4.3] for more details. By [Moo97, Proposition 2.6] model sets are Delone sets. □

We look now for a fundamental domain of $\mathbb{K}_\beta$ modulo the lattice $\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$. We define $h_{\mathfrak{p}} = \min\{v_{\mathfrak{p}}(x) : x \in V_{\mathbb{Z}}\}$, for every $\mathfrak{p} \mid (\beta)$.

Lemma 1.24. *Let $\{v_1, \ldots, v_d\}$ be a set of rationally independent generators of $V_{\mathbb{Z}}$. Then the set*

$$D = \left\{ \sum_{i=1}^{d} r_i \delta_\infty(v_i) : r_i \in [0, 1) \right\} \times \prod_{\mathfrak{p} \mid (\beta)} \mathfrak{p}^{h_{\mathfrak{p}}}$$

is a fundamental domain for $\mathbb{K}_\beta$ modulo $\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$.

Proof. Let $w_1, \ldots, w_d$ be an integral basis of $\mathcal{O}$ over $\mathbb{Z}$. We claim that the set

$$D_0 := \left\{ \sum_{i=1}^{d} r_i \delta_\infty(w_i) : r_i \in [0, 1) \right\} \times \prod_{\mathfrak{p} \mid (\beta)} \mathcal{O}_{\mathfrak{p}}$$

is a fundamental domain for $\mathbb{K}_\beta$ module $\delta(\mathcal{O}_S)$.

To prove this claim let $\mathbf{z} := (z_{\mathfrak{p}})_{\mathfrak{p} \in S} \in \mathbb{K}_\beta$. We know that $\delta_\infty(w_1), \ldots, \delta_\infty(w_d)$ is a basis of the real vector space $\mathbb{K}_\infty$. Thus $\mathbf{z}_\infty := (z_{\mathfrak{p}})_{\mathfrak{p} \mid \infty} = \sum_{i=1}^{d} r_i \delta_\infty(w_i) \in \mathbb{K}_\infty$ for some $r_i \in \mathbb{R}$, and we denote by $\iota(\mathbf{z}_\infty)$ the element $\sum_{i=1}^{d} \lfloor r_i \rfloor w_i \in \mathcal{O}$. For $\mathfrak{p} \mid (\beta)$, $z_{\mathfrak{p}} \in K_{\mathfrak{p}}$ can be written as

$$z_{\mathfrak{p}} = \sum_{i=-m}^{-1} c_i \nu^i + \sum_{i=0}^{\infty} c_i \nu^i, \qquad m \in \mathbb{N},$$

where ν is a uniformiser and the c_i are taken in a system of representatives of the residue class field $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$. Basically we view $z_{\mathfrak{p}}$ as the sum of a $\mathfrak{p}$-adic fractional part, that we denote by $\lambda_{\mathfrak{p}}(z_{\mathfrak{p}})$, and a $\mathfrak{p}$-adic integral part. Define

$$y = \sum_{\mathfrak{p} \mid (\beta)} \lambda_{\mathfrak{p}}(z_{\mathfrak{p}}) + \iota\left(\mathbf{z}_\infty - \delta_\infty\left(\sum_{\mathfrak{p} \mid (\beta)} \lambda_{\mathfrak{p}}(z_{\mathfrak{p}}) \right) \right).$$

One checks that $y \in \mathcal{O}_S$ and $\mathbf{z} - \delta(y) \in D_0$. Indeed, $\mathbf{z}_\infty - \delta_\infty(y)$ is an element of the form $\sum_{i=1}^{d} r_i \delta_\infty(w_i)$ with $r_i \in [0, 1)$, by definition of y, and, for $\mathfrak{p} \mid (\beta)$, observe that both $z_{\mathfrak{p}} - \delta_{\mathfrak{p}}(\sum_{\mathfrak{p} \mid (\beta)} \lambda_{\mathfrak{p}}(z_{\mathfrak{p}}))$ and $\delta_{\mathfrak{p}}(\iota(\mathbf{z}_\infty - \delta_\infty(\sum_{\mathfrak{p} \mid (\beta)} \lambda_{\mathfrak{p}}(z_{\mathfrak{p}}))))$ are in $\mathcal{O}_{\mathfrak{p}}$. Furthermore $\mathbf{z} - \delta(x) \notin D_0$ for all $x \in \mathcal{O}_S \setminus \{y\}$ (note that the intervals for the r_i in the definition of D_0 are half-open).

Now we replace the lattice $\delta(\mathcal{O}_S)$ with the sublattice $\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$. As $w_1, \ldots, w_d$ is a $\mathbb{Q}$-basis for K, the same holds for $v_1, \ldots, v_d$. The completion of $V_{\mathbb{Z}}$ at $\mathfrak{p}$, i.e., $(V_{\mathbb{Z}})_{\mathfrak{p}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is isomorphic to $\mathfrak{p}^{h_{\mathfrak{p}}}$. We can express an element $z_{\mathfrak{p}}$ of the completion $K_{\mathfrak{p}}$ as $z_{\mathfrak{p}} = \sum_{i=-m}^{-1} c_i \nu^i + \sum_{i=0}^{\infty} c_i \nu^i$ where ν is a uniformiser and the c_i are taken in a set of representatives of the residue class field $(V_{\mathbb{Z}})_{\mathfrak{p}}/\mathfrak{p}(V_{\mathbb{Z}})_{\mathfrak{p}}$ isomorphic to $\mathfrak{p}^{h_{\mathfrak{p}}}/\mathfrak{p}^{h_{\mathfrak{p}}+1}$. Thus we can adapt all the arguments given above to get a unique element $y \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}]$ such that $\mathbf{z} - \delta(y) \in D$. $\square$

LEMMA 1.25. $\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}]) = \beta \cdot \delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$.

PROOF. We know that $\beta V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \subset V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}]$, therefore $\beta \cdot \delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}]) \subset \delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$. The set $\beta \cdot \delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$ is a sublattice of $\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$. Let D' be a fundamental domain of $\mathbb{K}_\beta/\beta \cdot \delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$ and recall that D is a fundamental domain of $\mathbb{K}_\beta/\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$ (see Lemma 1.24). Then, by the product formula

$$\mu(D') = \mu(D) \prod_{\mathfrak{p} \in S} |\beta|_{\mathfrak{p}} = \mu(D),$$

and the claim follows. $\square$

1.3.2. Projections. There exists a unique M_σ-invariant decomposition $\mathbb{R}^n = V \oplus W$, where $M_\sigma|_V$ is hyperbolic (by the Pisot hypothesis) with characteristic polynomial $f(x)$. Let $\{\beta^{(1)}, \ldots, \beta^{(r)}, \beta^{(r+1)}, \overline{\beta^{(r+1)}}, \ldots, \beta^{(r+s)}, \overline{\beta^{(r+s)}}\}$ be the Galois conjugates of $\beta = \beta^{(1)}$. Choose dual bases $\{\mathbf{u}_{\beta^{(i)}}\}_{i=1}^d$, $\{\mathbf{v}_{\beta^{(i)}}\}_{i=1}^d$ and $\{\mathbf{u}_{\zeta^{(j)}}\}_{j=1}^{n-d}$, $\{\mathbf{v}_{\zeta^{(j)}}\}_{j=1}^{n-d}$ of right and left eigenvectors for M_σ associated with the Galois conjugates $\{\beta^{(i)}\}_{i=1}^d$ of $\beta = \beta^{(1)}$, respectively with the roots of the neutral polynomial $\{\zeta^{(j)}\}_{j=1}^{n-d}$, such that $n - d = r' + 2s'$, r' and $2s'$ denoting the number of real and complex roots of $g(x)$. We choose $\mathbf{v}_\beta \in \mathbb{Z}[\beta]^n$ and we renormalize $\mathbf{u}_\beta$ such that $\langle \mathbf{u}_\beta, \mathbf{v}_\beta \rangle = 1$. Notice that $\mathbf{v}_\beta \perp W$. Then, we can write $\mathbf{x} \in \mathbb{R}^n$ as

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_{\beta^{(1)}} \rangle \mathbf{u}_{\beta^{(1)}} + \sum_{i=2}^{r+s} \langle \mathbf{x}, \mathbf{v}_{\beta^{(i)}} \rangle \mathbf{u}_{\beta^{(i)}} + \sum_{j=1}^{r'+s'} \frac{\langle \mathbf{x}, \mathbf{v}_{\zeta^{(j)}} \rangle}{\langle \mathbf{v}_{\zeta^{(j)}}, \mathbf{u}_{\zeta^{(j)}} \rangle} \mathbf{u}_{\zeta^{(j)}}.$$

We will use especially in Chapter 2 and 4 in connection with the dual operators $\mathbf{E}_k^*(\sigma)$ the following projections:

$$\pi : \mathbb{R}^n \to \mathbb{K}_\beta, \quad \mathbf{x} \mapsto \left((\langle \mathbf{x}, \mathbf{v}_{\beta^{(i)}} \rangle)_{i=1}^{r+s}, (\langle \mathbf{x}, \mathbf{v}_\beta \rangle)_{\mathfrak{p}|(\beta)} \right) \tag{1.13}$$

$$\pi_e : \mathbb{R}^n \to \mathbb{K}_\beta^e, \quad \mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{v}_\beta \rangle \tag{1.14}$$

$$\pi_c : \mathbb{R}^n \to \mathbb{K}_\beta^c, \quad \mathbf{x} \mapsto \left((\langle \mathbf{x}, \mathbf{v}_{\beta^{(i)}} \rangle)_{i=2}^{r+s}, (\langle \mathbf{x}, \mathbf{v}_\beta \rangle)_{\mathfrak{p}|(\beta)} \right) \tag{1.15}$$

Let $\mathbf{v}_\beta = (v_1, \ldots, v_n)$. Then $\pi_e(\mathbf{e}_i) = v_i$, for all $i \in \mathcal{A}$.

PROPOSITION 1.26. *The following diagram*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{M_\sigma} & \mathbb{R}^n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{K}_\beta & \xrightarrow{\beta} & \mathbb{K}_\beta \end{array}$$

is commutative, where the action of β on $\mathbb{K}_\beta$ is defined by $\beta \cdot (\xi_{\mathfrak{p}})_{\mathfrak{p} \in S} = (\beta \xi_{\mathfrak{p}})_{\mathfrak{p} \in S}$.

PROOF. Use the fact that the $\mathbf{v}_{\beta^{(i)}}$ are left eigenvectors of M_σ associated with $\beta^{(i)}$, for $i = 1, \ldots, d$. $\square$

Since $M_\sigma \notin \mathrm{GL}_n(\mathbb{Z})$ if β is not a unit, we will consider in Chapter 2 the set $\mathscr{Z}_\sigma^n := \bigcup_{k \geq 0} M_\sigma^{-k} \mathbb{Z}^n$. Observe that $\delta \circ \pi_e = \pi$ and $\delta_c \circ \pi_e = \pi_c$ if we restrict the attention to $\mathscr{Z}_\sigma^n$. We will often use $\delta(v_i)$, $\delta_c(v_i)$ equivalently as $\pi(\mathbf{e}_i)$, $\pi_c(\mathbf{e}_i)$ respectively.

PROPOSITION 1.27. *$\pi(\mathscr{Z}_\sigma^n) = \delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$ and therefore is a lattice in $\mathbb{K}_\beta$.*

PROOF. $\pi(\mathscr{Z}_\sigma^n)$ coincides with $\delta(V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}])$ by Proposition 1.26, which is a lattice in $\mathbb{K}_\beta$ by Lemma 1.23. □

1.3.3. Approximation results. We list some results that are used in the proofs of Chapter 3.

LEMMA 1.28 (Strong Approximation Theorem, see e.g. [Cas67]). *Let S be a finite set of primes and let $\mathfrak{p}_0$ be a prime of a number field K which does not belong to S. Let $z_\mathfrak{p} \in K_\mathfrak{p}$ be given numbers, for $\mathfrak{p} \in S$. Then, for every $\varepsilon > 0$, there exists $x \in K$ such that*

$$|x - z_\mathfrak{p}|_\mathfrak{p} < \varepsilon \text{ for } \mathfrak{p} \in S, \text{ and } |x|_\mathfrak{p} \leq 1 \text{ for } \mathfrak{p} \notin S \cup \{\mathfrak{p}_0\}.$$

LEMMA 1.29. *For each $x \in \mathbb{Z}[\beta^{-1}] \setminus \mathbb{Z}[\beta]$, we have $\delta_\mathfrak{f}(x) \notin \overline{\delta_\mathfrak{f}(\mathbb{Z}[\beta])}$.*

PROOF. We first show that

$$\mathbb{Z}[\beta^{-1}] \cap \mathcal{O} \subseteq \beta^{-h}\mathbb{Z}[\beta] \tag{1.16}$$

for some $h \in \mathbb{N}$. As $\mathbb{Z}[\beta]$ is a subgroup of finite index of $\mathcal{O}$, we can choose $x_1, \ldots, x_m \in \mathcal{O}$ that form a complete set of representatives of $\mathcal{O}/\mathbb{Z}[\beta]$. Choose integers $h_1, \ldots, h_m$ as follows. If $x_i \notin \mathbb{Z}[\beta^{-1}]$, then set $h_i = 0$ and notice that $(x_i + \mathbb{Z}[\beta]) \cap \mathbb{Z}[\beta^{-1}] = \emptyset$. If $x_i \in \mathbb{Z}[\beta^{-1}]$, then choose $h_i \geq 0$ such that $x_i \in \beta^{-h_i}\mathbb{Z}[\beta]$, hence $x_i + \mathbb{Z}[\beta] \subseteq \beta^{-h_i}\mathbb{Z}[\beta]$. Then

$$\mathbb{Z}[\beta^{-1}] \cap \mathcal{O} = \bigcup_{i=1}^{m} \big(\mathbb{Z}[\beta^{-1}] \cap (\mathbb{Z}[\beta] + x_i)\big) \subseteq \beta^{-\max\{h_i\}}\mathbb{Z}[\beta],$$

thus (1.16) holds with $h = \max\{h_i\}$.

Let now $x \in \mathbb{Z}[\beta^{-1}] \setminus \mathbb{Z}[\beta]$ and suppose that $\delta_\mathfrak{f}(x) \in \overline{\delta_\mathfrak{f}(\mathbb{Z}[\beta])}$. Then there is $y \in \mathbb{Z}[\beta]$ such that $|y - x|_\mathfrak{p} \leq |\beta^h|_\mathfrak{p}$ for all $\mathfrak{p} \mid (\beta)$, with h as above, i.e., $y - x \in \beta^h\mathcal{O}$. By (1.16), we obtain that $y - x \in \mathbb{Z}[\beta]$, contradicting that $y \in \mathbb{Z}[\beta]$ and $x \notin \mathbb{Z}[\beta]$. □

Recall that if $\mathfrak{p}^e$ appear in the prime ideal factorization of $(p) = p\mathcal{O}$, then $e_{\mathfrak{p}|(p)} = e$ is called the ramification index and $f_{\mathfrak{p}|(p)} = [\mathcal{O}/\mathfrak{p} : \mathbb{Z}/p\mathbb{Z}]$ is the inertia degree of $\mathfrak{p} \mid (p)$ (see [Neu99]).

LEMMA 1.30. *Let $(\beta) = \prod_i \mathfrak{p}_i^{m_i}$, with $\mathfrak{p}_i \mid (p_i)$. Then $\overline{\delta_\mathfrak{f}(\mathbb{Q})} = \mathbb{K}_\mathfrak{f}$ if and only if $e_{\mathfrak{p}_i|(p_i)} = f_{\mathfrak{p}_i|(p_i)} = 1$ for all i and the prime numbers p_i are all distinct.*

If β is quadratic, $\beta^2 = a\beta + b$, then $\gcd(a,b) = 1$ implies $\overline{\delta_\mathfrak{f}(\mathbb{Q})} = \mathbb{K}_\mathfrak{f}$.

PROOF. By Lemma 1.28, $\delta_\mathfrak{f}(\mathbb{Q})$ is dense in $\prod_i \mathbb{Q}_{p_i}$ if and only if the p_i are distinct. By $[K_{\mathfrak{p}_i} : \mathbb{Q}_{p_i}] = e_{\mathfrak{p}_i|(p_i)} f_{\mathfrak{p}_i|(p_i)}$, if either $e_{\mathfrak{p}_i|(p_i)}$ or $f_{\mathfrak{p}_i|(p_i)}$ is greater than 1, then $\delta_\mathfrak{f}(\mathbb{Q})$ cannot be dense in $\mathbb{K}_\mathfrak{f}$. The other direction is similar.

If $\beta^2 = a\beta + b$ and $\gcd(a,b) = 1$, given $p \mid b$, we have that $p \nmid \operatorname{disc}(\mathbb{Z}[\beta]) = a^2 + 4b$. Thus $p \nmid [\mathcal{O} : \mathbb{Z}[\beta]]$, by the formula $\operatorname{disc}(\mathbb{Z}[\beta]) = [\mathcal{O} : \mathbb{Z}[\beta]]^2 \cdot \operatorname{disc}(\mathbb{Q}(\beta))$ (see e.g. [Coh93, Proposition 4.4.4]). Hence we can apply [Coh93, Theorem 4.8.13] and obtain that (p) splits, since $\gcd(a,b) = 1$. This means $e_{\mathfrak{p}|(p)} = f_{\mathfrak{p}|(p)} = 1$ for all $\mathfrak{p} \mid (p)$. □

Recall the definitions of L and property (QM) in Section 1.2.2.

Lemma 1.31. *The set $\delta(\mathbb{Z}[\beta])$ is a lattice in Z. If (QM) holds, then $\delta_c(L)$ is a lattice in Z^c.*

Proof. The sets $\delta_\infty(\mathbb{Z}[\beta])$ and, if (QM) holds, $\delta_\infty^c(L)$ are Delone subgroups in $\mathbb{K}_\infty$ and $\mathbb{K}_\infty^c$, respectively. Since $\overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$ is compact, we obtain that $\delta(\mathbb{Z}[\beta])$ and $\delta_c(L)$ are Delone subgroups in Z and Z^c, respectively. □

Lemma 1.32. *Assume that $\deg(\beta) \ge 2$. For each $y \in \mathbb{Z}[\beta]$, $k \in \mathbb{N}$, $\varepsilon > 0$, we have*

$$\frac{\#\big\{x \in \big(y + \beta^k\mathbb{Z}[\beta]\big) \cap [0,1) : \delta_\infty^c(x) \in X\big\}}{\#\big\{x \in \mathbb{Z}[\beta] \cap [0,1) : \delta_\infty^c(x) \in X\big\}} \le \frac{2+\varepsilon}{|N(\beta)|^k}$$

for each rectangle $X \subseteq \mathbb{K}_\infty^c$ with sufficiently large side lengths.

Proof. Let $k \in \mathbb{N}$, choose a set of representatives Y of $\mathbb{Z}[\beta]/\beta^k\mathbb{Z}[\beta]$ with $Y \subseteq [0,1)$, and set

$$C(y) = \#\big\{x \in \big(y + \beta^k\mathbb{Z}[\beta]\big) \cap [0,1) : \delta_\infty^c(x) \in X\big\}.$$

For $y, \tilde{y} \in Y$ with $y < \tilde{y}$, choose $z \in \mathbb{Z}[\beta]$ with $\tilde{y} - y \le \beta^k z \le 1$. (This is possible because $\beta^k\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$ by the irrationality of β.) Then

$$\begin{aligned}\big(y + \beta^k\mathbb{Z}[\beta]\big) \cap [0,1) \supseteq & \Big(\big(\tilde{y} + \beta^k\mathbb{Z}[\beta]\big) \cap [\tilde{y} - y, 1) + y - \tilde{y}\Big) \\ & \cup \Big(\big(\tilde{y} + \beta^k\mathbb{Z}[\beta]\big) \cap [0, \tilde{y} - y) + \beta^k z + y - \tilde{y}\Big),\end{aligned}$$

which implies the two inequalities

$$\begin{aligned}C(y) \ge & \#\big\{x \in \big(\tilde{y} + \beta^k\mathbb{Z}[\beta]\big) \cap [\tilde{y} - y, 1) : \delta_\infty^c(x) \in X\big\} \\ & - \#\big\{x \in \big(\tilde{y} + \beta^k\mathbb{Z}[\beta]\big) \cap [\tilde{y} - y, 1) : \delta_\infty^c(x) \in X,\, \delta_\infty^c(x + y - \tilde{y}) \notin X\big\}, \\ C(y) \ge & \#\big\{x \in \big(\tilde{y} + \beta^k\mathbb{Z}[\beta]\big) \cap [0, \tilde{y} - y) : \delta_\infty^c(x) \in X\big\} \\ & - \#\big\{x \in \big(\tilde{y} + \beta^k\mathbb{Z}[\beta]\big) \cap [0, \tilde{y} - y) : \delta_\infty^c(x) \in X,\, \delta_\infty^c(x + \beta^k z + y - \tilde{y}) \notin X\big\}.\end{aligned}$$

Since $\delta_\infty^c(\mathbb{Z}[\beta] \cap [0,1))$ is a Delone set by Lemma 1.23, the subtracted quantities are small compared to $C(\tilde{y})$, provided that $(X + \delta_\infty^c(y - \tilde{y})) \setminus X$ and $(X + \delta_\infty^c(\beta^k z + y - \tilde{y})) \setminus X$ are small compared to X. If X is a rectangle with sufficiently large side lengths, we have thus

$$2\,C(y) \ge \frac{C(\tilde{y})}{1 + \varepsilon/2}.$$

Similar arguments provide the same inequality for $y > \tilde{y}$, thus

$$C(y) \ge \frac{1}{2+\varepsilon} \max_{\tilde{y} \in Y} C(\tilde{y})$$

for all $y \in Y$. Summing over Y gives that

$$\#\big\{x \in \mathbb{Z}[\beta] \cap [0,1) : \delta_\infty^c(x) \in X\big\} \ge \frac{|N(\beta)|^k}{2+\varepsilon} \max_{\tilde{y} \in Y} C(\tilde{y}),$$

which proves the lemma. □

1.3.4. Representing p-adic spaces. We are interested in visualizing p-adic spaces through a Euclidean model (see e.g. [Rob00]) so that we will be able later to represent practically our non-unit Rauzy fractals. We can define the *Monna map*

$$\psi : \mathbb{Q}_p \to \mathbb{R}^+, \quad \sum_{i=m}^{\infty} d_i p^i \mapsto \sum_{i=m}^{\infty} d_i p^{-i-1}, \quad m \in \mathbb{Z}.$$

This map is onto, continuous, preserves the Haar measure but is not injective and not a homomorphism with respect to the addition. In our case we will represent elements of a $\mathfrak{p}$-adic completion $K_\mathfrak{p}$, isomorphic to a finite extension of $\mathbb{Q}_p$, for $\mathfrak{p} \mid (p)$, as $\sum_{i=m}^{\infty} d_i \xi^i$, where ξ is a uniformiser, i.e. $v_\mathfrak{p}(\xi) = 1$, and the d_i are chosen in a complete set of coset representatives of $\mathcal{O}/\mathfrak{p}\mathcal{O}$. By Lemma 1.28 we have that $\delta_\mathfrak{f}(\mathbb{Z}[\beta])$ is dense in $\mathbb{K}_\mathfrak{f}$, thus each element of $\mathbb{K}_\mathfrak{f}$ can be even written as $\sum_{i=m}^{\infty} \delta_\mathfrak{f}(d_i\beta^i)$, $d_i \in \{0, 1, \dots, |N(\beta)| - 1\}$. We can then apply the Monna map to each of the completions $K_\mathfrak{p}$ appearing in $\mathbb{K}_\mathfrak{f}$ and obtain

$$\mathbb{K}_\mathfrak{f} \to (\mathbb{R}^+)^\ell, \quad \sum_{i=m}^{\infty} \delta_\mathfrak{f}(d_i\beta^i) \mapsto \left(\sum_{i=m}^{\infty} d_i p^{-i-1} \right)_p$$

where ℓ is the number of primes $\mathfrak{p} \mid (\beta)$ and $\mathfrak{p} \mid (p)$.

1.4. Tilings

Let $\mathcal{C}$ be a collection of compact subsets of positive measure of a measurable space X.

- $\mathcal{C}$ is called *uniformly locally finite* if there exists an integer k such that each point of X is contained in at most k elements of $\mathcal{C}$.
- If moreover there exists a positive integer m such that almost every point of X is contained in exactly m elements of $\mathcal{C}$, then we call $\mathcal{C}$ a *multiple tiling* of X and m the *covering degree* of the multiple tiling.
- If $m = 1$, then $\mathcal{C}$ is called a *tiling* of X.
- A (multiple) tiling is called *periodic* if the translation set is a lattice. It is called *aperiodic* if it lacks any translational symmetry. It is *self-replicating* if there is an inflation factor α and a subdivision rule such that $\alpha\mathcal{C} = \mathcal{C}$.
- A point of X is called *exclusive point* of $\mathcal{C}$ if it is contained in exactly one element of $\mathcal{C}$. Thus, a multiple tiling is a tiling if and only if it has an exclusive point.

CHAPTER 2

The geometry of non-unit Pisot substitutions

In this chapter we present several approaches on how to define Rauzy fractals and stepped surfaces for non-unit Pisot substitutions and discuss the relations between them. In particular, we consider Rauzy fractals as the natural geometric objects of Dumont-Thomas numeration, in terms of the dual of the one-dimensional realization of the substitution, and in the context of model sets for particular cut and project schemes. We also define stepped surfaces suited for non-unit Pisot substitutions. We provide basic topological and geometric properties of Rauzy fractals, prove some tiling results for them, and provide relations to subshifts defined in terms of the periodic points of the substitution, to adic transformations, and a domain exchange. This chapter is based on [MT14].

2.1. Rauzy fractals and stepped surfaces

Rauzy fractals can be defined in several ways. In this section we give our main definition, related to Dumont-Thomas numeration, and we define stepped surfaces in the general settings of (non-unit) irreducible Pisot substitutions.

2.1.1. Dumont-Thomas tiles. Basic notions on Dumont-Thomas numeration, like σ-integers and σ-fractional parts, are presented in Section 1.2.1. We want to embed the σ-integers into $\mathbb{K}_\beta^c$ to obtain a geometrical representation of Dumont-Thomas numeration. Observe that

$$\overline{\delta_c(\operatorname*{Lim}_{k\to\infty} \mathbb{Z}_{\sigma,a}^{(k)})} = \operatorname*{Lim}_{k\to\infty} \delta_c(\mathbb{Z}_{\sigma,a}^{(k)}) = \lim_{k\to\infty} \delta_c(\mathbb{Z}_{\sigma,a}^{(k)}) = \lim_{k\to\infty} \delta_c(\beta^k\, T_\sigma^{-k}(0,a)), \tag{2.1}$$

where lim denotes the limit with respect to the Hausdorff metric and Lim is the topological limit. Indeed, the third equality holds since all $\delta_c(\mathbb{Z}_{\sigma,a}^{(k)})$ are contained in a compact set, and the fourth follows easily recalling the definition of $\mathbb{Z}_{\sigma,a}^{(k)}$.

Definition 2.1. Let σ be an irreducible Pisot substitution. The *Dumont-Thomas subtiles* associated with σ are defined as

$$\mathcal{R}_\sigma(a) = \lim_{k\to\infty} \delta_c(\beta^k\, T_\sigma^{-k}(0,a)) \quad \text{for} \quad a \in \mathcal{A}, \tag{2.2}$$

where the limit is taken with respect to the Hausdorff metric. The *Dumont-Thomas central tile* is defined as

$$\mathcal{R}_\sigma = \bigcup_{a\in\mathcal{A}} \mathcal{R}_\sigma(a). \tag{2.3}$$

Note that these tiles include those defined in [Aki02, ABBS08], since Dumont-Thomas numeration generalizes beta-numeration. An example of central tile is depicted in Figure 2.1.

For $\mathbf{z} = (z_{\mathfrak{p}})_{\mathfrak{p}\in S\setminus\{\mathfrak{p}_1\}} \in \mathbb{K}_\beta^c$ define the norm $\|\mathbf{z}\| = \max\{|z_{\mathfrak{p}}|_{\mathfrak{p}} : \mathfrak{p} \in S \setminus \{\mathfrak{p}_1\}\}$, and set

$$M = \frac{\max\{\|\delta_c(v_p)\| : v_p \in \mathcal{D}\}}{1 - \|\delta_c(\beta)\|}. \tag{2.4}$$

Note that the Dumont-Thomas subtiles $\mathcal{R}_\sigma(a)$ are closed by definition. Furthermore they are contained in the closed ball $B(\mathbf{0}, M) = \{\mathbf{z} \in \mathbb{K}_\beta^c : \|\mathbf{z}\| \leq M\}$. Thus they are non-empty compact sets. We will prove more properties of these tiles later.

We define the *x-tiles* as

$$\mathcal{R}_x = \bigcup_{\{a\in\mathcal{A}:\, x\in[0,v_a)\}} \lim_{k\to\infty} \delta_c(\beta^k T_\sigma^{-k}(x,a)).$$

Using (1.6) we see easily that they are unions of subtiles translated by $\delta_c(x)$ that depend on the number of basic intervals which contain x, i.e.,

$$\mathcal{R}_x = \bigcup_{\{a\in\mathcal{A}:\, x\in[0,v_a)\}} \mathcal{R}_\sigma(a) + \delta_c(x).$$

In this chapter we will consider mainly subtiles. The x-tiles will be our main objects in Chapter 3.

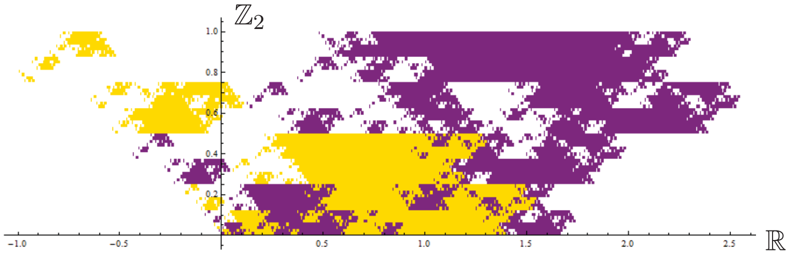


FIGURE 2.1. The central tile $\mathcal{R}_\sigma$ for $\sigma(1) = 2121^3$, $\sigma(2) = 12$ subdivided in the purple subtile $\mathcal{R}_\sigma(1)$ and the yellow subtile $\mathcal{R}_\sigma(2)$.

2.1.2. Stepped surfaces. We define stepped surfaces for irreducible Pisot substitutions in full generality, without requiring the Pisot substitution to be unit.

DEFINITION 2.2. The *stepped surface* for an irreducible Pisot substitution σ with associated Pisot root β is

$$\mathcal{S} = \{(\delta(x), a) \in \mathbb{K}_\beta \times \mathcal{A} : x \in \mathrm{Frac}(\sigma, a)\}. \tag{2.5}$$

The set of projected points of the stepped surface into $\mathbb{K}_\beta^c$ given by

$$\Gamma = \{(\delta_c(x), a) \in \mathbb{K}_\beta^c \times \mathcal{A} : x \in \mathrm{Frac}(\sigma, a)\} \tag{2.6}$$

will be called the *translation set*.

Notice that it makes sense to call Γ a translation set since it is a Delone set by Lemma 1.23. As we will see later in Section 2.4, this set is the natural translation set for a (multiple) tiling induced by the subtiles.

For our purposes (particularly in Section 2.5) we will interpret $(\gamma, a) \in \mathbb{K}_\beta \times \mathcal{A}$ either as a coloured translation point or as a coloured face of the fundamental domain $\mathbb{K}_\beta/\delta(V_\mathbb{Z}\cdot\mathbb{Z}[\beta^{-1}])$ described in Lemma 1.24. To be more precise, in this latter case, (γ, a) will be represented as $\gamma + F_a$, where

$$F_a = \left\{ \sum_{i\neq a} r_i\delta_\infty(v_i) : r_i \in [0,1) \right\} \times \prod_{\mathfrak{p}|(\beta)} \mathfrak{p}^{h_\mathfrak{p}},$$

and $h_\mathfrak{p} = \min\{v_\mathfrak{p}(x) : x \in V_\mathbb{Z}\}$. This construction appeared already in [Sin06b, Section 6.8] for the non-unit case.

The set function T_σ^{-1} defined in (1.5) is defined on subsets of $\mathbb{R}\times\mathcal{A}$. Its restriction to subsets of $\mathbb{Q}(\beta)\times\mathcal{A}$ admits a natural extension to $\mathbb{K}_\beta\times\mathcal{A}$. We denote this extension by $\mathbf{T}_\sigma^{-1}$. Its precise definition is $\mathbf{T}_\sigma^{-1}\circ\delta = \delta\circ T_\sigma^{-1}|_{\mathbb{Q}(\beta)\times\mathcal{A}}$.

$$\text{(2.7)}\quad \mathbf{T}_\sigma^{-1} : \delta(\mathbb{Q}(\beta))\times\mathcal{A} \to 2^{\delta(\mathbb{Q}(\beta))\times\mathcal{A}}, \quad \mathbf{T}_\sigma^{-1}(\gamma, a) = \bigcup_{b\xrightarrow{p}a} \{(\beta^{-1}(\gamma+\delta(v_p)), b)\}.$$

We can iterate $\mathbf{T}_\sigma^{-1}$ m times and get

$$\text{(2.8)}\quad \mathbf{T}_\sigma^{-m}(\gamma, a) = \bigcup_{\sigma^m(b)=pas} (\beta^{-m}(\gamma+\delta(v_p)), b).$$

PROPOSITION 2.3. *The set $\mathcal{S}$ is invariant under $\mathbf{T}_\sigma^{-1}$.*

PROOF. We prove first that if $(\delta(x), a) \in \mathcal{S}$ then $\mathbf{T}_\sigma^{-1}(\delta(x), a) \in \mathcal{S}$. This is equivalent in showing that every element of $T_\sigma^{-1}(x, a) \in \mathrm{Frac}(\sigma)$. But this is a direct consequence of Lemma 1.11.

Then we show that distinct faces have disjoint images. Suppose $(\delta(y), b) \in \mathbf{T}_\sigma^{-1}(\delta(x_1), a_1) \cap \mathbf{T}_\sigma^{-1}(\delta(x_2), a_2)$, that is

$$(y, b) \in T_\sigma^{-1}(x_1, a_1) \cap T_\sigma^{-1}(x_2, a_2).$$

This implies that $T_\sigma(y, b) = (x_1, a_1)$ and $T_\sigma(y, b) = (x_2, a_2)$, which is impossible unless $(x_1, a_1) = (x_2, a_2)$, since y has a unique (σ, b)-expansion. □

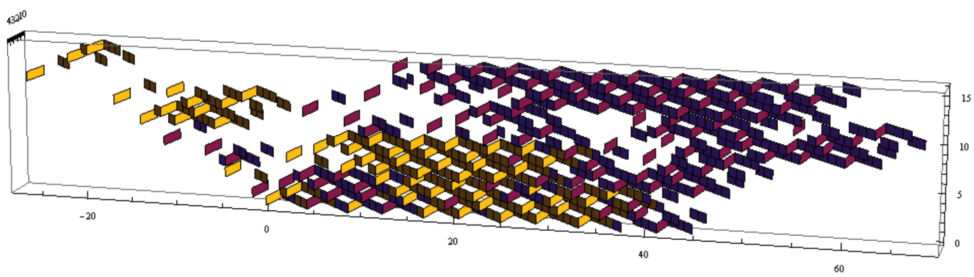


FIGURE 2.2. $\mathbf{T}_\sigma^{-4}((\mathbf{0},1)\cup(\mathbf{0},2))$ for the substitution $\sigma(1) = 2121^3$, $\sigma(2) = 12$.

2.2. Relations between different approaches

We give alternative definitions for the Rauzy fractals and we investigate their relations.

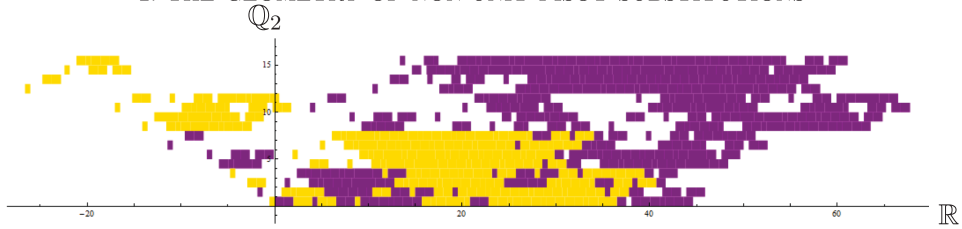


FIGURE 2.3. $\mathbf{T}_\sigma^{-4}((\mathbf{0},1)\cup(\mathbf{0},2))$ projected into $\mathbb{K}_\beta^c$.

2.2.1. Broken lines. A *broken line* is the geometrical interpretation of a fixed point u of σ:

$$\overline{u} = \bigcup_{i\geq 1}\{(\mathbf{l}(u_{[0,i)}), u_i)\}, \tag{2.9}$$

where $u_{[0,i)} = u_0\cdots u_{i-1}$ and $(\mathbf{x}, i)$ denotes the segment from $\mathbf{x}$ to $\mathbf{x}+\mathbf{e}_i$. We show that Rauzy fractals can be seen as projections of vertices of broken lines (cf. [BS05, Theorem 5]).

We will use the equivalent reformulation of Definition 2.1 of Dumont-Thomas tiles

$$\mathcal{R}_\sigma(a) = \Big\{\sum_{i\geq 0}\delta_c(v_{p_i}\beta^i) : (p_i)_{i\geq 0}\in\mathcal{G}_p(a)\Big\}, \tag{2.10}$$

where $\mathcal{G}_p(a)$ denotes the set of labels of infinite paths in the prefix graph of the substitution ending at a.

PROPOSITION 2.4. *We have*

$$\mathcal{R}_\sigma(a) = \overline{\{\pi_c(\mathbf{l}(u_{[0,i)})) : i\in\mathbb{N}, u_i = a)\}}, \quad a\in\mathcal{A}. \tag{2.11}$$

PROOF. We saw already in (1.3) that every prefix $u_{[0,i)}$ of u with $u_i = a$ can be expanded as $\sigma^k(p_k)\cdots\sigma(p_1)p_0$, where $\sigma(u_0) = p_k a_k s_k$, $p_k\neq\epsilon$, and $(p_i)_{i=0}^k$ is a walk in the prefix graph of σ starting from u_0 and ending at $a = a_0$. Observe that $\pi_c(\mathbf{l}(\sigma(p))) = \pi_c(M_\sigma\mathbf{l}(p)) = \delta_c(\beta v_p)$, and from the equivalent definition (2.10) of Dumont-Thomas subtiles the inclusion $\supseteq$ follows. For the other inclusion we can assume without loss of generality that M_σ is positive. Thus, every $(p_i)_{i=0}^k\in\mathcal{G}_p(a)$ can be extended to $(p_i)_{i=0}^{k+1}\in\mathcal{G}_p(a)$, such that $\sigma(u_0)$ contains a_{k+1} and $\sigma(a_{k+1}) = p_k a_k s_k$. We conclude by observing that every sum $\sum_{i\geq 0}\delta_c(v_{p_i}\beta^i)$, $(p_i)_{i\geq 0}\in\mathcal{G}_p(a)$ can be expressed as $\lim_{k\to\infty}\pi_c(\mathbf{l}(\sigma^k(p_k)\cdots\sigma(p_1)p_0))$. □

2.2.2. Projective limit. For general notions on projective limits we refer to [RZ10]. For $a\in\mathcal{A}$, let us consider the sets

$$\mathbb{Z}_{\sigma,a}^{(i)} = \beta^i\, T_\sigma^{-i}(0,a), \quad i\geq 1,$$

which can be thought as *approximations* of the set of integers $\mathbb{Z}_{\sigma,a}$. The sets $\mathbb{Z}_{\sigma,a}^{(i)}$ together with the continuous maps $f_i^j : \mathbb{Z}_{\sigma,a}^{(j)}\to\mathbb{Z}_{\sigma,a}^{(i)}$, $\sum_{k=0}^{j-1}v_{p_k}\beta^k\mapsto\sum_{k=0}^{i-1}v_{p_k}\beta^k$, for $i,j\in\mathbb{N}^+$ and $j\geq i$, form a projective system. Consider the projective limit

$$\hat{\mathbb{Z}}_{\sigma,a} = \varprojlim_i \mathbb{Z}_{\sigma,a}^{(i)} = \Big\{(x_i)_{i=1}^\infty\in\prod_{i\geq 1}\mathbb{Z}_{\sigma,a}^{(i)} :\ \text{for all } j\geq i,\ f_i^j(x_j) = x_i\Big\}, \tag{2.12}$$

and the union $\hat{\mathbb{Z}}_\sigma = \bigcup_{a\in\mathcal{A}}\hat{\mathbb{Z}}_{\sigma,a}$.

If we give to each $\mathbb{Z}_{\sigma,a}^{(i)}$ the discrete topology and to $\prod_{i\geq 1}\mathbb{Z}_{\sigma,a}^{(i)}$ the product topology, the space $\hat{\mathbb{Z}}_{\sigma,a}$ inherits a topology which turns it into a compact space.

We can equip $\hat{\mathbb{Z}}_{\sigma,a}$ with two other topologies. Indeed, equip $\hat{\mathbb{Z}}_{\sigma,a}$ with the topology defined by the distance

$$d(x,y) = 2^{-\max\{m\in\mathbb{N}:\, x_m=y_m\}}, \quad \text{for } x=(x_i)_{i\geq 1},\, y=(y_i)_{i\geq 1}\in\hat{\mathbb{Z}}_{\sigma,a}.$$

In this way $\hat{\mathbb{Z}}_{\sigma,a}$ is a compact Cantor set isomorphic to the subshift formed by the left-infinite sequences $(v_{p_i})_{i\geq 0}\in {}^{\omega}\mathcal{D}$ such that $(p_i)_{i\geq 0}$ is the labelling of a left-infinite walk $\cdots \xrightarrow{p_2} a_2 \xrightarrow{p_1} a_1 \xrightarrow{p_0} a$ in the prefix graph of σ.

On the other hand we can equip $\hat{\mathbb{Z}}_{\sigma,a}$ with the topology defined by the distance $d(x,y)=\|\delta_c(x)-\delta_c(y)\|$. An element of $\hat{\mathbb{Z}}_{\sigma,a}$ can be represented as an infinite sum $\sum_{i\geq 0} v_{p_i}\beta^i$, where each truncation $\sum_{i=0}^{\ell} v_{p_i}\beta^i$ is contained in $\mathbb{Z}_{\sigma,a}^{(\ell+1)}$. Extending δ_c continuously, we can use this mapping in order to map the infinite sums in $\hat{\mathbb{Z}}_{\sigma,a}$ to $\mathbb{K}_\beta^c$. Therefore we have obtained the following equivalence with the Dumont-Thomas subtiles.

Proposition 2.5. *We have* $\mathcal{R}_\sigma(a)=\delta_c(\hat{\mathbb{Z}}_{\sigma,a})$, *for* $a\in\mathcal{A}$.

The definition of the projective limit $\hat{\mathbb{Z}}_{\sigma,a}$ encompasses these two points of view, thus we can consider one of its elements either as an admissible left infinite sequence or as an infinite sum. Another advantage is that we have all the approximations (i.e., the truncations) of the elements included in this vision. In both interpretations multiplication by β acts as a contraction. We come back to $\hat{\mathbb{Z}}_{\sigma,a}$ in Section 2.3.2.

2.2.3. Dual substitutions. A consequence of dealing with non-unit Pisot substitutions is that $M_\sigma\notin \mathrm{GL}_n(\mathbb{Z})$. However it is invertible on

$$\mathscr{Z}_\sigma^n = \bigcup_{k\geq 0} M_\sigma^{-k}\mathbb{Z}^n.$$

We denote by $\mathcal{F}$ the infinite dimensional real vector space of the maps $\mathscr{Z}_\sigma^n\times\mathcal{A}\to\mathbb{R}$ that take value zero except for a finite set. For $(\mathbf{x},a)\in\mathscr{Z}_\sigma^n\times\mathcal{A}$ denote by $[\mathbf{x},a]$ the element of $\mathcal{F}$ which takes value 1 at $(\mathbf{x},a)$ and 0 elsewhere; the set $\{[\mathbf{x},a] : (\mathbf{x},a)\in\mathscr{Z}_\sigma^n\times\mathcal{A}\}$ is a basis of $\mathcal{F}$. The support of an element of $\mathcal{F}$ is the set of $[\mathbf{x},a]$ on which it is not zero.

We define the *one-dimensional geometric realization* $\mathbf{E}_1(\sigma)$ on $\mathcal{F}$ by

$$\mathbf{E}_1(\sigma)[\mathbf{y},b] = \sum_{b\xrightarrow{p}a} [M_\sigma\mathbf{y}-\mathbf{l}(p),a].$$

Denote by $\mathcal{F}^*$ the space of linear forms on $\mathcal{F}$ with finite support, i.e., those linear forms for which there exists a finite subset X of $\mathscr{Z}_\sigma^n\times\mathcal{A}$ such that the form is 0 on any element of $\mathcal{F}$ whose support does not intersect X; this space admits as basis the set $\{[\mathbf{x},a]^* : (\mathbf{x},a)\in\mathscr{Z}_\sigma^n\times\mathcal{A}\}$. We can associate with $\mathbf{E}_1(\sigma)$ its *dual map* $\mathbf{E}_1^*(\sigma)$ on $\mathcal{F}^*$.

Proposition 2.6. *The following formula for the dual map* $\mathbf{E}_1^*(\sigma)$ *holds:*

$$\mathbf{E}_1^*(\sigma)[\mathbf{x},a]^* = \sum_{b\xrightarrow{p}a} [M_\sigma^{-1}(\mathbf{x}+\mathbf{l}(p)),b]^*. \tag{2.13}$$

Proof. By definition of the dual map we have

$$\begin{aligned}\langle \mathbf{E}_1^*(\sigma)[\mathbf{x},a]^*, [\mathbf{y},b]\rangle &= \langle [\mathbf{x},a]^*, \mathbf{E}_1(\sigma)[\mathbf{y},b]\rangle \\ &= \langle [\mathbf{x},a]^*, \sum_{b \xrightarrow{p} c} [M_\sigma \mathbf{y} - \mathbf{l}(p), c]\rangle.\end{aligned}$$

This product can take only 0 and 1 as values and is not zero if and only if $c = a$ and $M_\sigma \mathbf{y} - \mathbf{l}(p) = \mathbf{x}$. Since M_σ is invertible as a map from $\mathscr{Z}_\sigma^n$ to $\mathscr{Z}_\sigma^n$, this implies $\mathbf{y} = M_\sigma^{-1}(\mathbf{x} + \mathbf{l}(p))$. $\square$

We can interpret geometrically an element $[\mathbf{x}, a] \in \mathcal{F}$ as a segment $\{\mathbf{x} - t\mathbf{e}_a : t \in [0,1]\}$ in $\mathbb{R}^n$. In this way we get the negative broken line $\overline{u}$ defined in (2.9) associated with the fixed point $u = u_0 u_1 \cdots$ of the substitution in terms of $\mathbf{E}_1(\sigma)$:

$$-\overline{u} = \bigcup_{k \geq 0} \mathbf{E}_1(\sigma)^k [\mathbf{0}, u_0]. \tag{2.14}$$

From now on we will only consider elements of $\mathcal{F}^*$ that are of the form $\sum_k [\mathbf{x}_k, a_k]^*$ (all the coefficients will be 1). Thus we shall consider $\mathbf{E}_1^*(\sigma)$ as a transformation acting directly on subsets of $\mathscr{Z}_\sigma^n \times \mathcal{A}$.

Lemma 2.7. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathscr{Z}_\sigma^n \times \mathcal{A} & \xrightarrow{\mathbf{E}_1^*(\sigma)} & 2^{\mathscr{Z}_\sigma^n \times \mathcal{A}} \\ \downarrow \pi & & \downarrow \pi \\ \delta(\mathbb{Q}(\beta)) \times \mathcal{A} & \xrightarrow{\mathbf{T}_\sigma^{-1}} & 2^{\delta(\mathbb{Q}(\beta)) \times \mathcal{A}} \end{array}$$

where π is the projection given in (1.13)

$$\pi : \mathbb{R}^n \to \mathbb{K}_\beta, \quad \mathbf{x} \mapsto \big((\langle \mathbf{x}, \mathbf{v}_{\beta^{(i)}} \rangle)_{i=1}^{r+s}, (\langle \mathbf{x}, \mathbf{v}_\beta \rangle)_{\mathfrak{p}|(\beta)} \big),$$

and by convention $\pi[\mathbf{x}, a]^*$ equals $(\pi(\mathbf{x}), a)$.

Proof. By Proposition 1.26 the action of M_σ on $\mathbb{R}^n$ is conjugate under π to multiplication by β on $\mathbb{K}_\beta$, which implies that $\pi(\mathscr{Z}_\sigma^n) = \delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])$. A simple computation using the definitions (2.7) of $\mathbf{T}_\sigma^{-1}$ and (2.13) of $\mathbf{E}_1^*(\sigma)$ shows that $\mathbf{T}_\sigma^{-1} \circ \pi = \pi \circ \mathbf{E}_1^*(\sigma)$. $\square$

Proposition 2.8. *The Dumont-Thomas subtiles can be expressed as*

$$\mathcal{R}_\sigma(a) = \lim_{k \to \infty} \pi_c(M_\sigma^k \, \mathbf{E}_1^*(\sigma)^k [\mathbf{0}, a]^*),$$

where the limit is taken with respect to the Hausdorff metric.

Proof. Simple consequence of Lemma 2.7. $\square$

Denote by $\mathbb{H}$ the hyperplane of $\mathbb{R}^n$ orthogonal to $\mathbf{v}_\beta$, and let $\mathbb{H}^+$ be the set $\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{v}_\beta \rangle \geq 0\}$, i.e., the half-space above $\mathbb{H}$. The half-space $\mathbb{H}^-$ strictly below $\mathbb{H}$ is defined in the same fashion. We look for all those $\mathbf{x} \in \mathscr{Z}_\sigma^n$ that are close to the hyperplane $\mathbb{H}$, in particular, we want $\mathbf{x} \in \mathbb{H}^+$ and $\mathbf{x} - \mathbf{e}_a \in \mathbb{H}^-$ to be true for some $a \in \mathcal{A}$. In the non-unit case we get too many points with this property, hence we project them by π in order to distribute them discretely with respect to their $\mathfrak{p}$-adic height.

Define the set of nearest coloured points to $\mathbb{H}$ in the spirit of [AI01] as

$$\Sigma = \{(\mathbf{x}, a) \in \mathscr{Z}_\sigma^n \times \mathcal{A} : \mathbf{x} \in \mathbb{H}^+, \ \mathbf{x} - \mathbf{e}_a \in \mathbb{H}^-\}.$$

Proposition 2.9. $\mathcal{S} = \pi(\Sigma)$.

Proof. As in the proof of Lemma 2.7 we have $\pi(\mathscr{Z}_\sigma^n) = \delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])$. The conditions $\mathbf{x} \in \mathbb{H}^+$, $\mathbf{x} - \mathbf{e}_a \in \mathbb{H}^-$ translate under π to $\delta(0) \leq \delta(x) < \delta(v_a)$, where $x = \langle \mathbf{x}, \mathbf{v}_\beta \rangle$. □

2.2.4. Model sets. We describe Sing's approach [Sin06b] for Rauzy fractals from the view point of cut-and-project schemes and model sets (see e.g. [Moo97, BM04, BG13]).

Definition 2.10. A *cut and project scheme*, or *CPS*, is a triple $(G, H, \tilde{L})$ consisting of a locally compact group G which is the union of countably many compact sets, called the *physical space*, a locally compact group H called the *internal space* and a lattice $\tilde{L}$ in $G \times H$, such that two natural projections $\pi_1 : G \times H \to G$, $\pi_2 : G \times H \to H$ satisfy the following properties:

(1) The restriction $\pi_1|_{\tilde{L}}$ is injective.
(2) The image $\pi_2(\tilde{L})$ is dense in H.

Setting $L = \pi_1(\tilde{L})$, the *star-map* is defined as $(\cdot)^\star = \pi_2 \circ (\pi_1|_{\tilde{L}})^{-1} : L \to H$, and is well-defined on L by injectivity of $\pi_1|_{\tilde{L}}$. With these definitions, we have $\tilde{L} = \{(x, x^\star) : x \in L\}$. We say that a cut and project scheme $(G, H, \tilde{L})$ is *symmetric* if $(H, G, \tilde{L})$ is a cut and project scheme as well. Given a cut and project scheme $(G, H, \tilde{L})$ and a subset $W \subset H$ define $\Lambda(W) = \{x \in L : x^\star \in W\}$. We call such a set $\Lambda(W)$, or more generally any translate of such a set, a *model set* if W is a non-empty compact set and $W = \overline{\mathrm{int}(W)}$. We say that a model set is *regular* if ∂W has zero Haar measure. In addition, we say that a set Q is an *inter model set* if $\Lambda(\mathrm{int}(W)) \subset Q \subset \Lambda(W)$.

A finite family $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ is a *multi-component Delone set* if $\mathrm{supp}(\underline{\Lambda}) = \bigcup_{a=1}^n \Lambda_a$ is a Delone set. Similarly we say that $\underline{\Lambda}$ is a *multi-component model set* if each $\Lambda_a = \Lambda(W_a)$ is a model set with respect to the same CPS.

Proposition 2.11. $(\mathbb{R}, \mathbb{K}_\beta^c, \delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]))$ *forms a symmetric cut and project scheme:*

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\pi_1} & \mathbb{K}_\beta & \xrightarrow{\pi_{S\setminus\{\mathfrak{p}_1\}}} & \mathbb{K}_\beta^c = \prod_{\mathfrak{p} \in S\setminus\{\mathfrak{p}_1\}} K_\mathfrak{p} \\
\cup & & \cup & & \cup \\
V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}] & \overset{1-1}{\longleftrightarrow} & \delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]) & \overset{1-1}{\longleftrightarrow} & \delta_c(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])
\end{array}$$

Proof. The set $\delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])$ is a lattice by Lemma 1.23. The projections π_1 and $\pi_{S\setminus\{\mathfrak{p}_1\}}$ are injective on $\delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])$ by construction. By Kronecker's theorem $V_\mathbb{Z}$ is dense in $\mathbb{R}$ and so is $V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]$. It remains to prove that $\delta_c(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])$ is dense in $\mathbb{K}_\beta^c$ (see [Sin06b, Lemma 6.55]). Since $\delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])$ is a lattice in $\mathbb{K}_\beta$, it is relatively dense. Hence $\delta_c(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}])$ must be relatively dense in $\mathbb{K}_\beta^c$, i.e., there exists a radius $R > 0$ such that $B(0, R) + \delta_c(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]) = \mathbb{K}_\beta^c$. Multiplying this equation by β (which is equivalent to a contraction in $\mathbb{K}_\beta^c$) and by Lemma 1.25 we get the denseness. □

For (Y, d) metric space, let $\mathcal{H}(Y)$ be the space of non-empty compact subsets of Y, equipped with the Hausdorff metric. In the model set setting, Sing [Sin06b] associates to each primitive substitution σ an *expanding matrix function system* $\boldsymbol{\Theta}$ on $\mathbb{R}^n$ defined by

$$\Theta_{ab} = \bigcup_{b \xrightarrow{p} a} \{t_{v_p} \circ f_0\}, \quad \text{for } a, b \in \mathcal{A}, \tag{2.15}$$

where $f_0(x) = \beta x$ and $t_{v_p}(x) = x + v_p$. Then its incidence matrix $\boldsymbol{S\Theta} := (|\Theta_{ab}|)_{a,b\in\mathcal{A}}$ equals M_σ. Given such a $\boldsymbol{\Theta}$ we can define the *adjoint iterated function system* $\boldsymbol{\Theta}^\#$ on $\mathcal{H}(\mathbb{R})^n$ by

$$\Theta^\#_{ab} = \bigcup_{a \xrightarrow{p} b} \{f_0^{-1} \circ t_{v_p}\}, \quad \text{for } a, b \in \mathcal{A}. \tag{2.16}$$

Then obviously $\boldsymbol{S\Theta}^\# = M_\sigma^t$. Note that $\boldsymbol{\Theta}^\#$ is just a way to write a *graph directed iterated function system* in the sense of Mauldin and Williams [MW88] in matrix form. By the general theory of graph directed iterated function systems there exists a unique attractor for $\boldsymbol{\Theta}^\#$, and it is easy to see that it is $\underline{A} = (A_a)_{a\in\mathcal{A}} \subset \mathcal{H}(\mathbb{R})^n$, where the $A_a = [0, v_a]$ are called *natural intervals*.

Geometrically we can interpret σ as a *tiling of the line*: given a fixed point $u = u_0u_1 \cdots \in \mathcal{A}^\omega$ of σ, we represent each letter a by the "type a" interval A_a; starting with the first of these intervals we can construct the entire line inflating repetitively A_a by β and subdividing it into the corresponding intervals given by the substitution (compare this to the action of the one-dimensional geometric realization $\mathbf{E}_1(\sigma)$ defined in Section 2.2.3; in particular, we refer to (2.14)).

Given the tiling of the line, denote the set of left endpoints of the type a intervals by Λ_a. Precisely, define $\underline{\Lambda} = (\Lambda_a)_{a\in\mathcal{A}}$ by

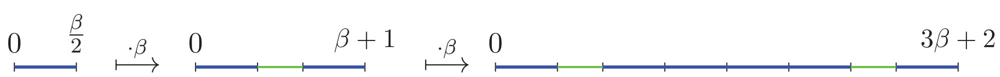
$$\underline{\Lambda} = \bigcup_{k\geq 0} \boldsymbol{\Theta}^k(\emptyset, \cdots, \emptyset, \{0\}, \emptyset, \cdots, \emptyset)^t, \tag{2.17}$$

where $\{0\}$ is at position u_0. Then $\underline{\Lambda} = (\Lambda_a)_{a\in\mathcal{A}}$ is a *substitution multi-component Delone set*, i.e., $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ and together with $\underline{A} = \boldsymbol{\Theta}^\#(\underline{A})$ this forms the *representation with natural intervals* $\underline{\Lambda} + \underline{A}$ of a fixed point u of σ.[1]

Example 2.12. Consider the substitution of Example 1.12, $\sigma(1) = 121$, $\sigma(2) = 11$. Then we obtain the following expanding matrix function system $\boldsymbol{\Theta}$ and its adjoint iterated function system $\boldsymbol{\Theta}^\#$:

$$\boldsymbol{\Theta} = \begin{pmatrix} \{f_0, f_{v_{12}}\} & \{f_0, f_{v_1}\} \\ \{f_{v_1}\} & \emptyset \end{pmatrix}, \qquad \boldsymbol{\Theta}^\# = \begin{pmatrix} \{g_0, g_{v_{12}}\} & \{g_{v_1}\} \\ \{g_0, g_{v_1}\} & \emptyset \end{pmatrix},$$

where $f_d(x) = \beta x + d$ and $g_d(x) = \beta^{-1}(x + d)$, for $d \in \mathcal{D} = \{0, v_1, v_{12}\}$. We get the tiling of the line applying repetitively the process of inflation and subdivision on the interval $[0, v_1]$:



[1] We could have defined $\boldsymbol{\Theta}$ using the functions $t_{-v_p} \circ f_0$, in accord with $\mathbf{E}_1(\sigma)$. In this way, we would have obtained a negative tiling of the line $-(\underline{\Lambda} + \underline{A})$, with Λ_a set of right endpoints of the type a intervals.

Furthermore we have that the sets $\varLambda_a$ of left endpoints of the type a intervals, for $a \in \mathcal{A}$, satisfy the point set equations

$$\varLambda_1 = \beta\varLambda_1 \cup (\beta\varLambda_1 + v_{12}) \cup \beta\varLambda_2 \cup (\beta\varLambda_2 + v_1),$$
$$\varLambda_2 = \beta\varLambda_1 + v_1,$$

and the natural intervals satisfy

$$A_1 = \beta^{-1}A_1 \cup \beta^{-1}(A_1 + v_{12}) \cup \beta^{-1}(A_2 + v_1),$$
$$A_2 = \beta^{-1}A_1 \cup \beta^{-1}(A_1 + v_1).$$

We can extend $\boldsymbol{\Theta}$ to the graph directed iterated function system $\boldsymbol{\Theta}^\star$ on $\mathcal{H}(\mathbb{K}^c_\beta)^n$ relative to the CPS $(\mathbb{R}, \mathbb{K}^c_\beta, \delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]))$ with star-map δ_c. As done before we can consider the adjoint $(\boldsymbol{\Theta}^\star)^\#$ on $(\mathbb{K}^c_\beta)^n$ relative to the CPS $(\mathbb{R}, \mathbb{K}^c_\beta, \delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]))$, which is an expanding matrix function system. This can now be used to define Rauzy fractals in this context.

Definition 2.13. Let $\underline{\varOmega} = (\varOmega_a)_{a\in\mathcal{A}} \subset \mathcal{H}(\mathbb{K}^c_\beta)^n$ be the solution of the graph directed iterated function system $\boldsymbol{\Theta}^\star(\underline{\varOmega}) = \underline{\varOmega}$. We call $\underline{\varOmega}$ the *dual prototile*. The regular multi-component inter model set $\underline{\varUpsilon} = (\varUpsilon_a)_{a\in\mathcal{A}}$ in $(\mathbb{K}^c_\beta)^n$ associated with the CPS $(\mathbb{K}^c_\beta, \mathbb{R}, \delta(V_\mathbb{Z} \cdot \mathbb{Z}[\beta^{-1}]))$, defined by $\varUpsilon_a = \Lambda([0, v_a))$, is called *translation set*.

We want now to relate the Dumont-Thomas tiles to the dual prototiles.

We can refine the sets $\mathbb{Z}^{(k)}_{\sigma,a}$ defined in Section 1.2.1 by taking only those finite integers associated with walks in the prefix graph starting at a state b and ending at state a. Call these sets $\mathbb{Z}^{(k)}_{b,a}$. By definition $\mathbb{Z}^{(k)}_{\sigma,a} = \bigcup_{b\in\mathcal{A}} \mathbb{Z}^{(k)}_{b,a}$. Moreover, if the word $\sigma(b)$ starts with b we easily see that the sequence $\big(\mathbb{Z}^{(k)}_{b,a}\big)_{k\geq 0}$ is nested.

Lemma 2.14. *Let $u = u_0u_1\cdots$ be the fixed point of σ and let $\underline{\varLambda}$ be as in (2.17). Then we have $(\mathbb{Z}_{u_0,a})_{a\in\mathcal{A}} = \underline{\varLambda}$. Furthermore $(\mathbb{Z}_{\sigma,a})_{a\in\mathcal{A}} = \mathrm{Lim}_{k\to\infty}\, \boldsymbol{\Theta}^k(\{0\})_{a\in\mathcal{A}}$, and in particular $(\mathbb{Z}_{\sigma,a})_{a\in\mathcal{A}} = \boldsymbol{\Theta}(\mathbb{Z}_{\sigma,a})_{a\in\mathcal{A}}$.*

Proof. We have $\boldsymbol{\Theta}^k(\emptyset, \ldots, \emptyset, \{0\}, \emptyset, \ldots, \emptyset)^t = (\Theta^k_{a,u_0}(\{0\}))_{a\in\mathcal{A}}$, whose elements are of the form $\beta^k v_{p_k} + \cdots + v_{p_0}$ where $u_0 \xrightarrow{p_k} \cdots \xrightarrow{p_0} a$. But these are elements of $\mathbb{Z}^{(k)}_{u_0,a}$, thus we have shown $(\mathbb{Z}_{u_0,a})_{a\in\mathcal{A}} = \underline{\varLambda}$. Recalling that $\mathbb{Z}^{(k)}_{\sigma,a} = \bigcup_{u_0\in\mathcal{A}} \mathbb{Z}^{(k)}_{u_0,a}$ we get the second statement. □

We are now in position to state the equivalence between Dumont-Thomas tiles and dual prototiles.

Proposition 2.15. *We have $\mathcal{R}_\sigma(a) = \varOmega_a$, for $a \in \mathcal{A}$.*

Proof. Recall that $\underline{\varOmega} = (\varOmega_a)_{a\in\mathcal{A}}$ is the attractor of the graph directed iterated function system $\boldsymbol{\Theta}^\star$, and the Dumont-Thomas subtiles can be defined as $\mathcal{R}_\sigma(a) = \overline{\delta_c(\mathbb{Z}_{\sigma,a})}$, see (2.1). Then by Lemma 2.14

$$(\mathcal{R}_\sigma(a))_{a\in\mathcal{A}} = \boldsymbol{\Theta}^\star(\overline{\delta_c(\mathbb{Z}_{\sigma,a})})_{a\in\mathcal{A}} = \boldsymbol{\Theta}^\star(\mathcal{R}_\sigma(a))_{a\in\mathcal{A}}$$

and the result follows by uniqueness of the attractor of $\boldsymbol{\Theta}^\star$. □

Proposition 2.16. *For $a \in \mathcal{A}$ we have $\varUpsilon_a = \delta_c(\mathrm{Frac}(\sigma, a))$, which shows that the translation set $\Gamma = \mathrm{supp}(\underline{\varUpsilon})$.*

Proof. Observe that

$$\Upsilon_a = \{\pi_{S\setminus\{\mathfrak{p}_1\}}(\mathbf{z}) \in \mathbb{K}_\beta^c : \mathbf{z} = (z_\mathfrak{p})_{\mathfrak{p}\in S} \in \delta(V_\mathbb{Z}\cdot\mathbb{Z}[\beta^{-1}]), z_{\mathfrak{p}_1} \in [0, v_a)\}$$

but this is exactly $\delta_c(\mathrm{Frac}(\sigma, a))$, for each $a \in \mathcal{A}$. □

2.3. Basic properties of the tiles

In this section we will present some basic topological and dynamical properties of the Rauzy fractals. By abuse of notation we will denote the projection of the action of $\mathbf{T}_\sigma^{-1}$ on $\delta_c(\mathbb{Q}(\beta)) \times \mathcal{A}$ again by $\mathbf{T}_\sigma^{-1}$. Then we can write the Rauzy fractals as

$$\mathcal{R}_\sigma(a) = \lim_{k\to\infty} \beta^k \cdot \mathbf{T}_\sigma^{-k}(\mathbf{0}, a). \tag{2.18}$$

2.3.1. Topological properties. The following proposition gives information on the $\mathfrak{p}$-adic height of the central tile.

Proposition 2.17. *If $\mathbf{z} = (z_\mathfrak{p}) \in \mathcal{R}_\sigma$, for every $\mathfrak{p} \mid (\beta)$ we have $z_\mathfrak{p} \in \mathfrak{p}^{h_\mathfrak{p}}$, where $h_\mathfrak{p} = \min\{v_\mathfrak{p}(x) : x \in V_\mathbb{Z}\}$.*

Proof. If $\mathbf{z} \in \mathcal{R}_\sigma$, we can write $\mathbf{z} = \sum_{k=0}^\infty \delta_c(v_{p_k}\beta^k)$. For $\mathfrak{p} \mid (\beta)$, the $\mathfrak{p}$-th component of $\mathbf{z}$ is $z_\mathfrak{p} = \sum_{k=0}^\infty v_{p_k}\beta^k \in K_\mathfrak{p}$. We deduce that $v_\mathfrak{p}(z_\mathfrak{p}) \geq \min_{k\geq 0}\{v_\mathfrak{p}(v_{p_k}\beta^k)\} = \min_{v_p\in\mathcal{D}}\{v_\mathfrak{p}(v_p)\} = d_\mathfrak{p}$, thus $z_\mathfrak{p} \in \mathfrak{p}^{h_\mathfrak{p}}$. □

Next we prove that the subtiles cover the representation space (cf. [ABBS08] and Chapter 3 for the non-unit beta-expansion setting).

Proposition 2.18. *Let σ be an irreducible Pisot substitution. The subtiles $\mathcal{R}_\sigma(a)$ provide a uniformly locally finite covering of the representation space $\mathbb{K}_\beta^c$ governed by Γ:*

$$\mathbb{K}_\beta^c = \bigcup_{(\gamma,a)\in\Gamma} \mathcal{R}_\sigma(a) + \gamma.$$

Proof. Let $\mathcal{C}_\sigma = \bigcup_{(\gamma,a)\in\Gamma} \mathcal{R}_\sigma(a) + \gamma$. Every point of $\mathcal{C}_\sigma$ is of the form $\mathbf{z} + \delta_c(x)$, where $\mathbf{z} = \sum_{i\geq 0}\delta_c(v_{p_i}\beta^i) \in \mathcal{R}_\sigma(a)$, $x = \sum_{i\geq 1} v_{p_{-i}}\beta^{-i} \in \mathrm{Frac}(\sigma, a)$. We have $\beta\cdot\mathcal{C}_\sigma \subseteq \mathcal{C}_\sigma$ since $T_\sigma(x, a) \in \mathrm{Frac}(\sigma, b)$ by Lemma 1.11 and $\beta\cdot\mathbf{z} + \delta_c(v_{p_{-1}}) \in \mathcal{R}_\sigma(b)$, for some $b \in \mathcal{A}$. By Lemma 1.23 $\mathcal{C}_\sigma$ is relatively dense in $\mathbb{K}_\beta^c$. Furthermore, as $\beta\cdot\mathcal{C}_\sigma \subseteq \mathcal{C}_\sigma$ and β is a contraction, $\mathcal{C}_\sigma$ is dense in $\mathbb{K}_\beta^c$. By compactness of the subtiles and uniformly discreteness of Γ we obtain $\mathbb{K}_\beta^c = \mathcal{C}_\sigma$. □

In the following theorem we state some important properties of our tiles (cf. [Sin06b, Corollary 6.66]).

Theorem 2.19. *The following assertions hold for the subtiles $\mathcal{R}_\sigma(a)$, $a \in \mathcal{A}$, of an irreducible Pisot substitution.*

(i) The subtiles $\mathcal{R}_\sigma(a)$ are the solution of the graph directed iterated function system

$$\mathcal{R}_\sigma(a) = \bigcup_{(\gamma,b)\in\mathbf{T}_\sigma^{-1}(\mathbf{0},a)} \beta\cdot(\mathcal{R}_\sigma(b) + \gamma) = \bigcup_{b\xrightarrow{p}a} \beta\cdot\mathcal{R}_\sigma(b) + \delta_c(v_p), \tag{2.19}$$

where the union is measure disjoint.

(ii) Each subtile $\mathcal{R}_\sigma(a)$ is the closure of its interior.

(iii) The boundary of each subtile $\mathcal{R}_\sigma(a)$ has Haar measure zero.

Proof. (i) Equation (2.19) is a direct consequence of Lemma 2.14, but we prefer to give here an explicit proof. By (2.18) and (1.6) we obtain

$$\begin{aligned}\mathcal{R}_\sigma(a) &= \lim_{k\to\infty} \beta^k \cdot \mathbf{T}_\sigma^{-k}(\mathbf{0},a) = \beta \lim_{k\to\infty} \bigcup_{(\gamma,b)\in\mathbf{T}_\sigma^{-1}(\mathbf{0},a)} \beta^{k-1}\cdot \mathbf{T}_\sigma^{-(k-1)}(\gamma,b)\\ &= \beta \cdot \bigcup_{(\gamma,b)\in\mathbf{T}_\sigma^{-1}(\mathbf{0},a)} (\mathcal{R}_\sigma(b)+\gamma) = \bigcup_{b\xrightarrow{p}a} \beta\cdot\mathcal{R}_\sigma(b)+\delta_c(v_p).\end{aligned}$$

Let $\mathbf{m} = (\mu_c(\mathcal{R}_\sigma(a)))_{a\in\mathcal{A}}$. Applying the measure μ_c to equation (2.19) gives

$$\begin{aligned}\mu_c(\mathcal{R}_\sigma(a)) &\le \sum_{b\xrightarrow{p}a} \mu_c(\beta\cdot\mathcal{R}_\sigma(b)+\delta_c(v_p))\\ &= \beta^{-1}\sum_{b\xrightarrow{p}a}\mu_c(\mathcal{R}_\sigma(b)) = \beta^{-1}\sum_{b\in\mathcal{A}}(M_\sigma)_{ab}\,\mu_c(\mathcal{R}_\sigma(b)).\end{aligned} \tag{2.20}$$

So we showed that the vector $\mathbf{m}$ satisfies $M_\sigma\mathbf{m} \ge \beta\mathbf{m}$, and, as a direct consequence of the Perron-Frobenius Theorem, we get $M_\sigma\mathbf{m} = \beta\mathbf{m}$. Thus the inequality in (2.20) is actually an equality, and thus no overlap with positive measure occurs in the union in (2.19).

(ii) Since $\mathbb{K}_\beta^c$ is locally compact, we deduce from Baire's theorem that there exists $a\in\mathcal{A}$ such that $\operatorname{int}(\mathcal{R}_\sigma(a)) \ne \emptyset$. Therefore (2.19) and the primitivity of σ yield that $\operatorname{int}(\mathcal{R}_\sigma(a)) \ne \emptyset$ holds for each $a\in\mathcal{A}$. Let now $a\in\mathcal{A}$ and consider $\eta\in\mathcal{R}_\sigma(a)$. Let B be an open ball centred at η. It suffices to show that $B\cap\operatorname{int}(\mathcal{R}_\sigma(a)) \ne \emptyset$. Using the k-fold iteration

$$\mathcal{R}_\sigma(a) = \bigcup_{(\gamma,b)\in\mathbf{T}_\sigma^{-k}(\mathbf{0},a)} \beta^k\cdot(\mathcal{R}_\sigma(b)+\gamma) \tag{2.21}$$

of (2.19) for k large enough, we obtain that $\beta^k\cdot(\mathcal{R}_\sigma(b)+\gamma)\subseteq B$ holds for some $(\gamma,b)\in\mathbf{T}_\sigma^{-k}(\mathbf{0},a)$. As $\operatorname{int}(\beta^k\cdot(\mathcal{R}_\sigma(b)+\gamma))\ne\emptyset$ the ball B contains inner points of $\mathcal{R}_\sigma(a)$.

(iii) Let $B\subset\operatorname{int}(\mathcal{R}_\sigma(a))$ be an open ball and fix $b\in\mathcal{A}$. By the primitivity of σ we may choose $k\in\mathbb{N}$ large enough such that $U := \beta^k\cdot(\mathcal{R}_\sigma(b)+\gamma)\subseteq B$ holds for some $(\gamma,b)\in\mathbf{T}_\sigma^{-k}(\mathbf{0},a)$. The boundary ∂U is a subset of the set that is covered at least twice by the union (2.21). We claim that $\mu_c(\partial U)=0$. Indeed, if $\mu_c(\partial U)>0$ was true, then

$$\mu_c(\mathcal{R}_\sigma(a)) \le \sum_{(\gamma,b)\in\mathbf{T}_\sigma^{-k}(\mathbf{0},a)} \mu_c(\beta^k\cdot(\mathcal{R}_\sigma(b)+\gamma)) - \mu_c(\partial U),$$

contradicting the measure disjointness of the union (2.21). Thus $\mu_c(\partial U)=0$ and, hence, $\mu_c(\partial\mathcal{R}_\sigma(b))=0$. Since $b\in\mathcal{A}$ was arbitrary, we are done. □

2.3.2. Adic transformation and domain exchange. Siegel [Sie03] showed that, if σ satisfies the strong coincidence condition (see Definition 1.3), the following hold:

(1) The subtiles $\mathcal{R}_\sigma(a)$ are disjoint in measure.
(2) (X_σ, S) is isomorphic in measure to $(\mathcal{R}_\sigma, E)$, where E is the *domain exchange* $E(\mathbf{z}) = \mathbf{z}+\delta_c(v_a)$, for $\mathbf{z}\in\mathcal{R}_\sigma(a)$.

We give a proof of the second result connecting it also to the *adic transformation* $\hat{\mathbb{Z}}_\sigma \to \hat{\mathbb{Z}}_\sigma$, $x \mapsto x + v_{w_0}$ on $\hat{\mathbb{Z}}_\sigma$ (cf. [CS01b, Proposition 2.3]).

We will need the following lemma (see [CS01a, Lemma 4.1, Proposition 5.1, Theorem 5.1]).

LEMMA 2.20. *Let $w \in X_\sigma$ and $\psi_{\mathcal{P}}(w) = (p_i, a_i, s_i)_{i\geq 0}$ its prefix-suffix development. Then $\psi_{\mathcal{P}}(\chi(w)) = (p_i, a_i, s_i)_{i\geq 1}$ and $\psi_{\mathcal{P}}(\sigma(w)) = (q_i, b_i, t_i)_{i\geq 0}$ is such that $q_0 = \epsilon$ and $q_{i+1} = p_i$, for every $i \geq 0$. If Sw is a periodic point of σ then $\psi_{\mathcal{P}}(Sw) = (\epsilon, b_i, t_i)_{i\geq 0}$ and $\psi_{\mathcal{P}}(w) = (p_i, a_i, \epsilon)_{i\geq 0}$, with $(p_i)_{i\geq 0}$ periodic. If Sw is not periodic for σ then $\psi_{\mathcal{P}}(Sw) = (q_i, b_i, t_i)_{i\geq 0}$ is such that there exists an integer k_0 with $\sigma^k(p_k)\cdots\sigma^0(p_0)a_0 = \sigma^k(q_k)\cdots\sigma^0(q_0)$, for all $k \geq k_0$.*

SKETCH OF THE PROOF. The first two statements follow from the definition of $\psi_{\mathcal{P}}$. For the third, a successor map ν defined on $X^l_{\mathcal{P}}$ and conjugate to the shift on X_σ is introduced, i.e., such that $\nu(\psi_{\mathcal{P}}(w)) = \psi_{\mathcal{P}}(Sw)$, for $w \in X_\sigma$. Given $(p_i, a_i, s_i)_{i\geq 0} \in X^l_{\mathcal{P}}$, $\nu((p_i, a_i, s_i)_{i\geq 0}) = (q_i, b_i, t_i)_{i\geq 0}$ is defined as follows: let i_0 be the first index such that $s_{i_0} \neq \epsilon$; then, $(q_{i_0}, b_{i_0}, t_{i_0})$ is such that $q_{i_0}b_{i_0}t_{i_0} = p_{i_0}a_{i_0}s_{i_0}$ and $|q_{i_0}| = |p_{i_0}| + 1$, for $i \geq i_0$, $(q_i, b_i, t_i) = (p_i, a_i, s_i)$, and for $0 \leq i < i_0$ we take (ϵ, b_i, t_i) such that $\sigma(b_{i+1}) = b_i t_i$. This is precisely an adic transformation, and, for its particular shape, we can deduce the last claim. □

PROPOSITION 2.21. *Let σ be an irreducible Pisot substitution satisfying the strong coincidence condition. Let*

$$\varphi : X_\sigma \to \hat{\mathbb{Z}}_\sigma, \quad w \mapsto \sum_{i\geq 0} v_{p_i}\beta^i$$

where $\psi_{\mathcal{P}}(w) = (p_i, a_i, s_i)_{i\geq 0} \in {}^\omega\mathcal{P}$ is the prefix-suffix development of $w = \cdots w_{-1}.w_0w_1\cdots$. Then the action of σ on X_σ is conjugate to the multiplication by β on $\hat{\mathbb{Z}}_\sigma$ and the following diagram

$$\begin{array}{ccccc}
X_\sigma & \xrightarrow{\varphi} & \hat{\mathbb{Z}}_\sigma & \xrightarrow{\delta_c} & \mathcal{R}_\sigma \\
{\scriptstyle S}\downarrow & & \downarrow{\scriptstyle +v_{w_0}} & & \downarrow{\scriptstyle +\delta_c(v_{w_0})} \\
X_\sigma & \xrightarrow{\varphi} & \hat{\mathbb{Z}}_\sigma & \xrightarrow{\delta_c} & \mathcal{R}_\sigma
\end{array}$$

is commutative.

PROOF. The first statement follows from Lemma 2.20 observing that the action of σ on X_σ is conjugate to the right extension of elements of $X^l_{\mathcal{P}}$ by an element that has an empty prefix.

The commutativity of the left diagram is also a consequence of Lemma 2.20. Let $w \in X_\sigma$ and $\psi_{\mathcal{P}}(w) = (p_i, a_i, s_i)_{i\geq 0}$. If Sw is not a periodic point of σ, $\psi_{\mathcal{P}}(Sw) = (q_i, b_i, t_i)_{i\geq 0}$ is such that $\exists\, k_0$ such that for all $k \geq k_0$ we have $\sigma^k(p_k)\cdots\sigma(p_0)w_0 = \sigma^k(q_k)\cdots\sigma^0(q_0)$. Thus

$$\varphi(Sw) = \sum_{i\geq 0} v_{q_i}\beta^i = \sum_{i\geq 0} v_{p_i}\beta^i + v_{w_0} = \varphi(w) + v_{w_0}.$$

If Sw is a periodic point of σ, then $\psi_{\mathcal{P}}(Sw) = (\epsilon, b_i, t_i)_{i\geq 0}$ and $\psi_{\mathcal{P}}(w) = (p_i, a_i, \epsilon)_{i\geq 0}$, with $(p_i)_{i\geq 0}$ periodic with period ℓ. This implies $\varphi(Sw) = 0$ and

$\sigma^{i+kl}(p_{i+k\ell})\cdots p_0w_0 = \sigma^{k\ell}(\sigma^i(p_i)\cdots p_0w_0)$ for every $i < \ell$ and every integer k. Therefore

$$\begin{aligned}\varphi(w) &= \lim_{k\to\infty}\sum_{i=0}^{k} v_{p_i}\beta^i = \lim_{k\to\infty}(v_{p_k}\beta^k + \cdots + v_{p_1}\beta + v_{p_0w_0}) - v_{w_0}\\ &= \lim_{k\to\infty}\beta^{k\ell}(v_{p_i}\beta^i + \cdots + v_{p_1}\beta + v_{p_0w_0}) - v_{w_0}\\ &= 0 - v_{w_0} = \varphi(Sw) - v_{w_0}.\end{aligned}$$

The commutativity of the right diagram follows simply applying δ_c extended to elements of $\hat{\mathbb{Z}}_\sigma$ and observing that the addition by $\delta_c(v_{w_0})$ is well-defined up to a set of measure zero. □

Observe that the adic transformation can be interpreted and computed by Bratteli diagrams (see e.g. [Dur10]).

2.4. Multiple tilings and property (F)

In this section we show that the subtiles $\mathcal{R}_\sigma(a)$ induce a multiple tiling of $\mathbb{K}_\beta^c$ with respect to the translation set Γ. Moreover, we give a tiling criterion in terms of a finiteness condition of (σ, a)-expansions.

2.4.1. Multiple tiling property. In this section we will prove that every irreducible Pisot substitution induces a multiple tiling of the associated representation space.

A *patch* is defined as a finite subset of Γ. We say that Γ is *repetitive* (or *quasi-periodic*) if for any patch P there exists a radius $R > 0$ such that every ball of radius R in Γ contains a translate of P.

Next result is already contained in [Sin06b, Proposition 6.72] in the model set approach. We present a similar proof in the spirit of [BST10, Theorem 5.3.13] in our setting.

Lemma 2.22. *The translation set Γ is repetitive.*

Proof. Let $P = \{(\gamma_k, a_k),\ 1 \le k \le \ell\}$ be a patch of Γ. We can write each γ_k as $\delta_c(x_k)$, for $x_k \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, v_{a_k})$. Let R_1 be such that $B(0, R_1)$ contains the patch P. There exists $\varepsilon_k > 0$ such that $x_k \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, (1-\varepsilon_k)v_{a_k})$, for each $1 \le k \le \ell$. Set $\varepsilon := \frac{1}{2}\min_k \varepsilon_k v_{a_k}$. Then $\delta_c(x) + P$ is in Γ, for every $x \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, \varepsilon)$.

It remains to prove that there exists $R > 0$ such that any ball of radius R in $\mathbb{K}_\beta^c$ contains a point $\delta_c(x)$ with $x \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, \varepsilon)$. By the denseness of $V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}]$ in $\mathbb{R}$ there exists $x_0 \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, \varepsilon/2)$. Let $\Delta = \max_{a\in\mathcal{A}} v_a$. We can divide $[0, \Delta)$ in $N = \lceil 2\Delta/\varepsilon \rceil$ subintervals $[j\varepsilon/2, (j+1)\varepsilon/2)$ of length $\varepsilon/2$. For each $j \le N$, there exists $m_j \in \mathbb{Z}$ such that $m_j x_0 + [j\varepsilon/2, (j+1)\varepsilon/2) \subset [0, \varepsilon)$.

Fix a point $\eta \in \mathbb{K}_\beta^c$. Since Γ is a Delone set we know that there is $R_2 > 0$ such that every ball of radius R_2 contains at least one element of Γ. In particular, the ball $B(\eta, R_2)$ contains a point $\delta_c(x)$ with $x \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap [0, \Delta)$. Thus there exists $j \in \{0, \dots, N\}$ such that $x \in V_{\mathbb{Z}} \cdot \mathbb{Z}[\beta^{-1}] \cap [j\varepsilon/2, (j+1)\varepsilon/2)$, and, hence, $m_j \in \mathbb{Z}$ such that $m_j x_0 + x \in [0, \varepsilon)$. This implies that $\delta_c(x + m_j x_0) + P$ occurs in Γ.

Therefore, the ball centred in η with radius $R := R_1 + R_2 + \max_j \|\delta_c(m_j x_0)\|$ contains a translated copy of the patch P and the lemma is proved. □

We are now in a position to state the multiple tiling result (see also [BST10, Theorem 5.3.13] for irreducible unit substitutions).

THEOREM 2.23. *Let σ be an irreducible Pisot substitution. The collection $\mathcal{C}_\sigma = \{\mathcal{R}_\sigma(a) + \gamma : (\gamma, a) \in \Gamma\}$ is a multiple tiling of $\mathbb{K}_\beta^c$.*

PROOF. Assume that the assertion of the theorem is false. Then there exist $\ell_1, \ell_2 \in \mathbb{N}$, $\ell_1 < \ell_2$, and $M_1, M_2 \subset \mathbb{K}_\beta^c$ with $\mu_c(M_i) > 0$ such that each element of M_i is covered exactly ℓ_i times by the elements of $\mathcal{C}_\sigma$ $(i = 1, 2)$. As the boundaries of the subtiles have zero measure by Theorem 2.19 ((iii)), there exist points $\eta_i \in M_i$ that are not contained in the boundary of any element of $\mathcal{C}_\sigma$. Thus we can find $\varepsilon > 0$ such that $B(\eta_i, \varepsilon)$ is covered exactly ℓ_i times by the collection $\mathcal{C}_\sigma$. This implies that there exists a patch $P_2 \subset \Gamma$ with ℓ_2 elements such that

$$B(\eta_2, \varepsilon) \subset \bigcap_{(\gamma,a)\in P_2} \mathcal{R}_\sigma(a) + \gamma.$$

Consider the inflated ball $\beta^{-k} \cdot B(\eta_1, \varepsilon)$. By the same arguments presented above, each point of $\beta^{-k} \cdot B(\eta_1, \varepsilon)$ is covered by exactly ℓ_1 tiles of the collection $\beta^{-k} \cdot \mathcal{C}_\sigma$. Each of the inflated tiles of $\beta^{-k} \cdot \mathcal{C}_\sigma$ can be decomposed in a finite union of tiles in $\mathcal{C}_\sigma$ which are pairwise disjoint in measure. Thus almost each point in $\beta^{-k} \cdot B(\eta_1, \varepsilon)$ is contained in exactly ℓ_1 tiles of $\mathcal{C}_\sigma$. By Lemma 2.22 we can pick a suitable large k such that $\beta^{-k} \cdot B(\eta_1, \varepsilon)$ contains a translated copy $P_2 + \gamma$, for some $\gamma \in \Gamma$. Therefore $B(\eta_2, \varepsilon) + \gamma$ is contained in $\beta^{-k} \cdot B(\eta_1, \varepsilon)$, for k large enough. The ball $B(\eta_2, \varepsilon)$ is covered exactly ℓ_2 times, consequently $B(\eta_2, \varepsilon) + \gamma$ is covered at least ℓ_2 times, but this yields a contradiction since almost every point in $\beta^{-k} \cdot B(\eta_1, \varepsilon)$ is contained in exactly ℓ_1 tiles, and $\ell_1 < \ell_2$. □

2.4.2. Finiteness property. In this section we provide a tiling criterion for $\mathcal{C}_\sigma$ based on the *geometric property* (F). We take inspiration mainly from [ST09]. Consider the set

$$\mathcal{U} := \bigcup_{a\in\mathcal{A}} (\mathbf{0}, a) \subset \Gamma.$$

It is easy to see that $\mathcal{U} \subseteq \mathbf{T}_\sigma^{-1}(\mathcal{U})$: indeed, $(\mathbf{0}, b) \in \mathbf{T}_\sigma^{-1}(\mathbf{0}, a)$ if $\sigma(b) = as$, i.e., if $p = \epsilon$. Thus $(\mathbf{0}, b) \in \mathbf{T}_\sigma^{-1}(\mathbf{0}, a)$ where a is the first letter of $\sigma(b)$. Hence the sequence $(\mathbf{T}_\sigma^{-m}(\mathcal{U}))_{m\geq 0}$ is an increasing sequence of subsets of Γ.

DEFINITION 2.24. Let σ be an irreducible Pisot substitution. We say that the substitution σ satisfies the *geometric property* (F) if the iterations of $\mathbf{T}_\sigma^{-1}$ on $\mathcal{U}$ eventually cover the whole self-replicating translation set Γ:

$$\Gamma = \bigcup_{m\geq 0} \mathbf{T}_\sigma^{-m}(\mathcal{U}).$$

The geometric property (F) is an equivalent formulation of the finiteness property firstly introduced in [FS92] in the beta-numeration framework and further studied in [Aki00]. Here we shall interpret it as a finiteness condition on (σ, b)-expansions. Indeed, given $(\gamma, b) \in \Gamma$, γ can be written as $\delta_c(x)$ where $x \in V \cdot \mathbb{Z}[\beta^{-1}] \cap [0, v_b)$ which has a unique (σ, b)-expansion by Proposition 1.5, i.e., $x = \sum_{i\geq 1} v_{p_i} \beta^{-i}$. Then we can say as well that (γ, b) has a formal (σ, b)-expansion in $\mathbb{K}_\beta^c$, namely $\gamma = \sum_{i\geq 1} \delta_c(v_{p_i} \beta^{-i})$.

PROPOSITION 2.25. *The substitution σ satisfies the geometric property (F) if and only if every point $(\gamma, b) \in \Gamma$ has a unique finite (σ, b)-expansion.*

PROOF. Let $(\gamma, b) \in \Gamma$. If property (F) holds, then $(\gamma, b) \in \mathbf{T}_\sigma^{-m}(\mathbf{0}, a)$, for some $m \geq 0$ and $a \in \mathcal{A}$. Thus, using (2.8) we get

$$\gamma = \delta_c(v_{p_0}\beta^{-m} + v_{p_1}\beta^{-m+1} + \cdots + v_{p_{m-1}}\beta^{-1}), \tag{2.22}$$

where $b \xrightarrow{p_{m-1}} \cdots \xrightarrow{p_1} a_1 \xrightarrow{p_0} a_0$ is a walk in the prefix graph ending at $a = a_0$.

On the other hand, suppose that $(\gamma, b) \in \Gamma$ has a unique finite (σ, b)-expansion $\gamma = \delta_c(v_{p_1}\beta^{-1} + \cdots + v_{p_m}\beta^{-m})$ with $b \xrightarrow{p_1} \cdots \xrightarrow{p_{m-1}} a_{m-1} \xrightarrow{p_m} a$. This yields that $\beta^m\gamma \in \mathcal{R}_\sigma(a)$ and using the iterated set equation in 2.21 we get

$$\gamma \in \bigcup_{(\eta,c)\in\mathbf{T}_\sigma^{-m}(\mathbf{0},a)} \mathcal{R}_\sigma(c) + \eta.$$

Thus we may conclude that $(\gamma, b) \in \mathbf{T}_\sigma^{-m}(\mathbf{0}, a)$. □

For an irreducible Pisot substitution σ satisfying the geometric property (F), it is immediate from Proposition 2.18 and the definition of subtiles that every $\mathbf{z} \in \mathbb{K}_\beta^c$ admits a (σ, a)-expansion in $\mathbb{K}_\beta^c$ for some $a \in \mathcal{A}$ (cf. [ST09, Proposition 3.9]), i.e.,

$$\mathbf{z} = \sum_{i=m}^{\infty} \delta_c(v_{p_i}\beta^i), \quad m \in \mathbb{Z}.$$

In the context of beta-numeration, Akiyama [Aki02] proved that property (F) is equivalent to the fact that $\mathbf{0}$ is an exclusive inner point of the central tile. Our next aim is to carry over this statement to the substitution context.

DEFINITION 2.26. The *zero-expansion graph* $\mathcal{G}^{(0)}$ of σ is the directed graph such that the following conditions hold.

- The nodes $(\gamma, a) \in \Gamma$ are such that $\|\gamma\| \leq M$, where M is taken as in Equation (2.4).
- There is a directed edge from (γ_1, a_1) to (γ_2, a_2) if and only if $(\gamma_2, a_2) \in \mathbf{T}_\sigma^{-1}(\gamma_1, a_1)$.
- Every node is the starting point of an infinite walk.

The zero-expansion graph is used to characterize all the elements $(\gamma, a) \in \Gamma$ for which the tile $\mathcal{R}_\sigma(a) + \gamma$ contains $\mathbf{0}$. Suppose $\mathbf{0} \in \mathcal{R}_\sigma(a) + \gamma$. This implies that $\gamma \in B(\mathbf{0}, M)$, where M is as in (2.4).

PROPOSITION 2.27. *The zero-expansion graph $\mathcal{G}^{(0)}$ of an irreducible Pisot substitution σ is well defined and finite. A pair (γ, a) is a node of this graph if and only if $\mathbf{0} \in \mathcal{R}_\sigma(a) + \gamma$.*

PROOF. The graph is finite since the nodes are elements of the Delone set Γ with bounded norm. Consider a node $(\gamma, a) = (\gamma_0, a_0) \in \mathcal{G}^{(0)}$ and the infinite walk $\{(\gamma_k, a_k)\}_{k\geq 0}$ starting from it. Then, by definition of edges, we get a left-infinite walk in the prefix graph $\cdots \xrightarrow{p_2} a_2 \xrightarrow{p_1} a_1 \xrightarrow{p_0} a_0$ and

$$\gamma = -\delta_c(v_{p_0} - v_{p_1}\beta - \cdots - v_{p_k}\beta^k) + \beta^{k+1} \cdot \gamma_{k+1}.$$

Since multiplication by β is a contraction and $\|\gamma_k\|$ is uniformly bounded in k, we obtain for $k \to \infty$ a convergent power series: $\gamma = -\sum_{k\geq 0} \delta_c(v_{p_k}\beta^k)$. Thus $-\gamma \in \mathcal{R}_\sigma(a)$ and hence $\mathbf{0} \in \mathcal{R}_\sigma(a) + \gamma$.

Suppose conversely that $\mathbf{0} \in \mathcal{R}_\sigma(a) + \gamma$, for $(\gamma, a) \in \Gamma$. Then

$$\gamma = -\sum_{k \geq 0} \delta_c(v_{p_k} \beta^k),$$

where $(p_k)_{k \geq 0}$ is the labelling of a left-infinite walk in the prefix graph ending at state a. Let $\gamma_\ell = -\sum_{k \geq 0} \delta_c(v_{p_{k+\ell}} \beta^k)$. Each $\gamma_\ell \in B(\mathbf{0}, M)$ and $\beta \cdot \gamma_{\ell+1} = \gamma_\ell + \delta_c(v_{p_\ell})$, i.e., $(\gamma_{\ell+1}, a_{\ell+1}) \in \mathbf{T}_\sigma^{-1}(\gamma_\ell, a_\ell)$. By induction $(\gamma_\ell, a_\ell) \in \Gamma$ for all $\ell \in \mathbb{N}$, since this holds for $\gamma = \gamma_0$ and Γ is invariant under $\mathbf{T}_\sigma^{-1}$ by Proposition 2.3. Hence, $(\gamma_k, a_k)_{k \geq 0}$ is an infinite walk in the zero-expansion graph starting from (γ, a). □

LEMMA 2.28. *Let σ be an irreducible Pisot substitution that satisfies the strong coincidence condition. Then σ satisfies the geometric property (F) if and only if $\mathbf{0}$ is an exclusive inner point of the central tile $\mathcal{R}_\sigma$.*

PROOF. Suppose that $\mathbf{0}$ is not an exclusive inner point of $\mathcal{R}_\sigma$. Then there exists $\gamma \neq \mathbf{0}$, which has a finite expansion by property (F), such that $\mathbf{0} \in \mathcal{R}_\sigma(a) + \gamma$, which implies $\mathbf{0} = \sum_{j=-m}^{\infty} \delta_c(v_{p_j} \beta^j)$, for $m \in \mathbb{N}$. Multiplying by β^{-k} yields $\mathbf{0} = \sum_{j=-m}^{\infty} \delta_c(v_{p_j} \beta^{j-k})$, that means $\mathbf{0} \in \mathcal{R}_\sigma(a_k) + \sum_{\ell=1}^{m+k} \delta_c(v_{p_{k-\ell}} \beta^{-\ell})$ for each $k \in \mathbb{N}$, where each of these sums represent a different element since the representation is unique. This gives a contradiction with the local finiteness of the covering. Therefore $\mathbf{0}$ is an exclusive inner point.

Assume that (F) does not hold, i.e., there exists $(\gamma_0, a_0) \in \Gamma \setminus \bigcup_{m \geq 0} \mathbf{T}_\sigma^{-m}(\mathcal{U})$. In particular $\gamma_0 \neq \mathbf{0}$. Since $\mathbf{T}_\sigma^{-1}(\Gamma) = \Gamma$, we can define a sequence $\{(\gamma_k, a_k)\}_{k \geq 1}$ of elements of Γ with

$$(\gamma_k, a_k) \in \mathbf{T}_\sigma^{-1}(\gamma_{k+1}, a_{k+1}), \quad k \geq 0.$$

Since multiplication by β is a contraction in $\mathbb{K}_\beta^c$, for some $k_0 \in \mathbb{N}$ large enough, $\gamma_k \in B(\mathbf{0}, M)$, for all $k \geq k_0$, where M is as in equation (2.4). There exist only finitely many $(\gamma_k, a_k) \in \Gamma$ such that $\gamma_k \in B(\mathbf{0}, M)$, since Γ is a Delone set. Then

$$\exists k' > k_0,\ \exists \ell > 0 \quad \text{such that} \quad (\gamma_{k'}, a_{k'}) = (\gamma_{k'+\ell}, a_{k'+\ell}),$$

and $\gamma_{k'} \neq \mathbf{0}$, otherwise $\gamma_0 \in \bigcup_{m \geq 0} \mathbf{T}_\sigma^{-m}(\mathcal{U})$. This is equivalent to the existence of a loop in the zero-expansion graph $\mathcal{G}^{(0)}$

$$\gamma_{k'} \to \gamma_{k'+\ell-1} \to \cdots \to \gamma_{k'+1} \to \gamma_{k'},$$

and, by the definition of $\mathcal{G}^{(0)}$, this implies that $\mathbf{0} \in \mathcal{R}_\sigma(a_{k'}) + \gamma_{k'}$. Since $\gamma_{k'} \neq \mathbf{0}$, we have that $\mathbf{0}$ is not an exclusive inner point of $\mathcal{R}_\sigma$. □

Finally we can generalize the tiling condition given in [ABBS08] for beta-numeration to (non-unit) irreducible Pisot substitutions.

THEOREM 2.29. *Let σ be an irreducible Pisot substitution. If σ satisfies the geometric property (F) and the strong coincidence condition, the self-replicating multiple tiling $\{\mathcal{R}_\sigma(a) + \gamma : (\gamma, a) \in \Gamma\}$ is a tiling.*

PROOF. By the geometric property (F) $\mathbf{0}$ is an exclusive inner point. Since the strong coincidence condition holds, the subtiles $\mathcal{R}_\sigma(a)$ are disjoint in measure (see the beginning of Section 2.3.2) and there is a set of positive measure around $\mathbf{0}$ which is covered only once. Since we know by Theorem 2.23 that $\{\mathcal{R}_\sigma(a) + \gamma : (\gamma, a) \in \Gamma\}$ is a multiple tiling, this implies that the covering degree is 1. □

2.5. Examples

In this section we consider two examples of irreducible non-unit Pisot substitutions.

2.5.1. A two letters example. Consider the substitution $\sigma(1) = 1^5 2$, $\sigma(2) = 1^3$. We have

$$M_\sigma = \begin{pmatrix} 5 & 3 \\ 1 & 0 \end{pmatrix}, \qquad \det(xI - M_\sigma) = x^2 - 5x - 3.$$

The dominant eigenvalue is $\beta = \frac{5+\sqrt{37}}{2}$ and its conjugate β' satisfies $|\beta'| < 1$.

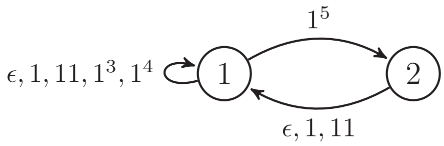


FIGURE 2.4. Prefix graph of σ.

The prime ideal $(3) = \mathfrak{p}_1 \mathfrak{p}_2 = (\beta)(5 - \beta)$ splits completely in $\mathcal{O}$. The normalized absolute values are such that $|\beta|_{\mathfrak{p}_1} = \frac{1}{3}$, $|\beta|_{\mathfrak{p}_2} = 1$, and we deduce that we have to consider only the non-Archimedean completion $K_{\mathfrak{p}_1}$, which is an extension of degree one of $\mathbb{Q}_3$, equipped with the normalized absolute value $|\cdot|_{\mathfrak{p}_1}$. Thus the representation space is $\mathbb{K}_\beta^c = \mathbb{R} \times \mathbb{Q}_3$. Notice that β is a uniformiser for $K_{\mathfrak{p}_1}$ and we can represent each element of $\mathbb{Q}_3$ as $\sum_{i=m}^{\infty} d_i \beta^i$ with $d_i \in \{0, 1, 2\}$, $m \in \mathbb{Z}$. Recall from Section 1.3.4 that we represent $\mathbb{Q}_3$ with the Euclidean model given by the Monna map. The canonical embedding is given explicitly by

$$\delta_c : \mathbb{Q}(\beta) \longrightarrow \mathbb{R} \times \mathbb{Q}_3, \qquad a_0 + a_1\beta \longmapsto \Big(a_0 + a_1\beta', \sum_{i=m}^{\infty} d_i \beta^i\Big).$$

We choose $\mathbf{v}_\beta = (\frac{\beta}{3}, 1)$ as left eigenvector of M_σ. The set of digits for the Dumont-Thomas expansions is $\mathcal{D} = \{0, v_1, 2v_1, 3v_1, 4v_1, 5v_1\}$. We obtain the central tile $\mathcal{R}_\sigma$ by taking the closure of the embedding of the σ-integers (see Figure 2.5). Having chosen $\mathbf{v}_\beta$ as above, we get that $\mathcal{R}_\sigma \subset \mathbb{R} \times \mathbb{Z}_3$ (see Proposition 2.17). Furthermore, the subtiles satisfy the following set equations:

$$\begin{aligned}
\mathcal{R}_\sigma(1) &= \beta \cdot \mathcal{R}_\sigma(1) + (\beta \cdot \mathcal{R}_\sigma(1) + \delta_c(v_1)) + (\beta \cdot \mathcal{R}_\sigma(1) + \delta_c(2v_1)) \\
&\quad + (\beta \cdot \mathcal{R}_\sigma(1) + \delta_c(3v_1)) + (\beta \cdot \mathcal{R}_\sigma(1) + \delta_c(4v_1)) \\
&\quad + \beta \cdot \mathcal{R}_\sigma(2) + (\beta \cdot \mathcal{R}_\sigma(2) + \delta_c(v_1)) + (\beta \cdot \mathcal{R}_\sigma(2) + \delta_c(2v_1)), \\
\mathcal{R}_\sigma(2) &= \beta \cdot \mathcal{R}_\sigma(1) + \delta_c(5v_1).
\end{aligned}$$

In Figure 2.6 a patch of the self-replicating tiling is illustrated. The first 3-adic level of the tiling, i.e., those tiles whose $\mathfrak{p}$-adic part is contained in $\mathbb{Z}_3$, consists in the $\mathcal{R}_x$ with

$$x \in \left\{\frac{2\beta}{3} - 3, \frac{2\beta}{3} - 2, \frac{\beta}{3} - 1, 0, 1, 2 - \frac{\beta}{3}, 3 - \frac{\beta}{3}\right\}.$$

One can check in fact that all these x we have $|x|_{\mathfrak{p}_1} \leq 1$. Furthermore the tiles $\mathcal{R}_x$, for $x = \frac{2\beta}{3} - 3, \frac{\beta}{3} - 1, 0, 2 - \frac{\beta}{3}$, are union of the two subtiles $\mathcal{R}_\sigma(1)$ and

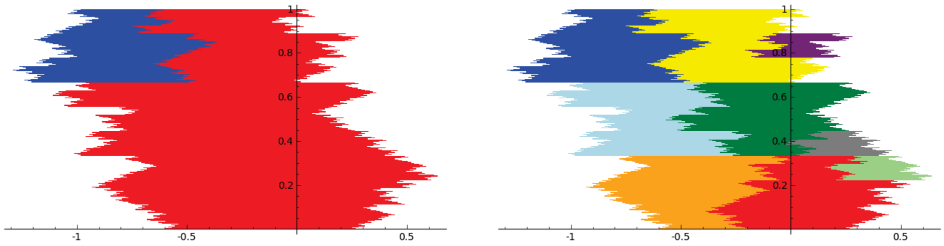


FIGURE 2.5. The central tile $\mathcal{R}_\sigma$ divided in the red (light gray) subtile $\mathcal{R}_\sigma(1)$ and the blue (dark gray) subtile $\mathcal{R}_\sigma(2)$, and the self similar structure arising from the set equations.

$\mathcal{R}_\sigma(2)$, because these x are less than 1, i.e., they are both in $[0, v_1) = [0, \frac{\beta}{3})$ and $[0, v_2) = [0, 1)$.

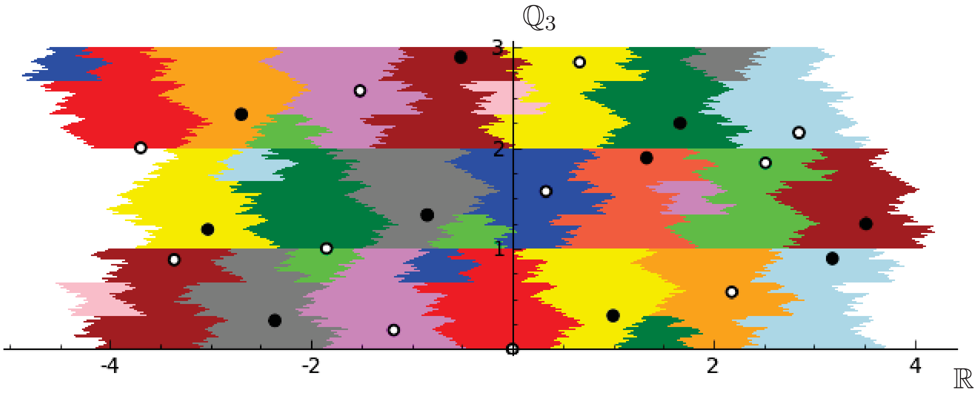
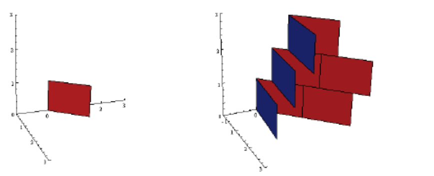
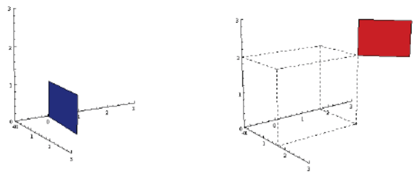


FIGURE 2.6. Tiling of the representation space $\mathbb{K}_\beta^c$ with translation set Γ. The black (white) points belong to $\delta_c(\mathrm{Frac}(\sigma, 1))$ (respectively $\delta_c(\mathrm{Frac}(\sigma, 2))$).

In Figure 2.7 and Figure 2.8 it is represented the action of $\mathbf{T}_\sigma^{-1}$ on the basic faces $(\mathbf{0}, 1)$, $(\mathbf{0}, 2)$ and on their union $\mathcal{U}$. This is an example of substitution satisfying the geometric property (F). Finally we show in Figure 2.9 the exchange of domains given by the substitution σ.



(A) The face $(\mathbf{0}, 1)$ and its image $\mathbf{T}_\sigma^{-1}(\mathbf{0}, 1)$.



(B) The face $(\mathbf{0}, 2)$ and its image $\mathbf{T}_\sigma^{-1}(\mathbf{0}, 2)$.

FIGURE 2.7

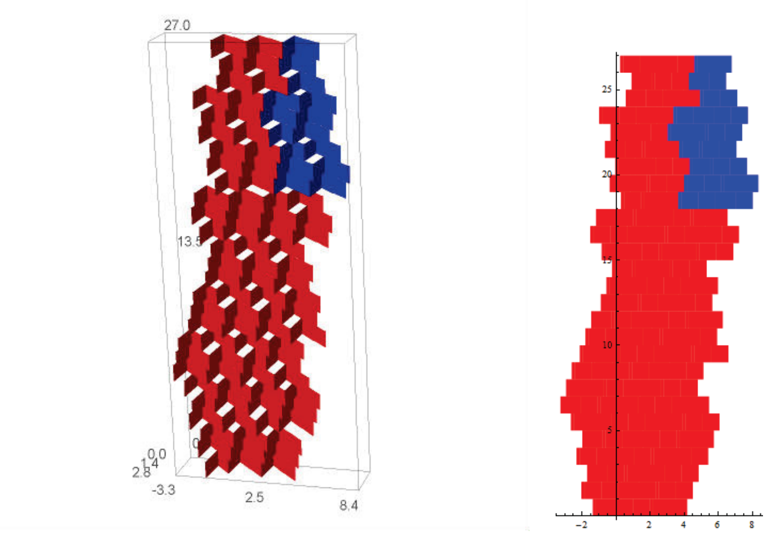


FIGURE 2.8. $\mathbf{T}_\sigma^{-3}(\mathcal{U})$ and its projection into $\mathbb{K}_\beta^c$.

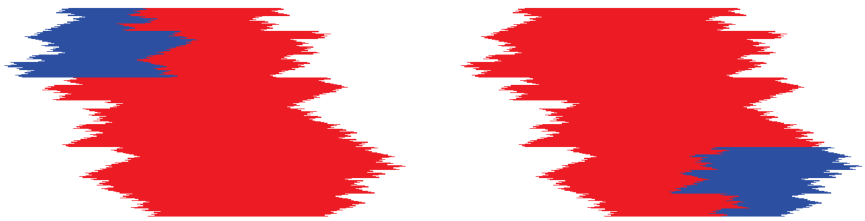


FIGURE 2.9. Exchange of domains.

2.5.2. A three letters example. Let σ be the substitution $\sigma : 1 \mapsto 1113$, $2 \mapsto 11$, $3 \mapsto 2$. The incidence matrix of the substitution and its characteristic polynomial are

$$M_\sigma = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \det(xI - M_\sigma) = x^3 - 3x^2 - 2.$$

The characteristic polynomial is irreducible over $\mathbb{Q}$ with Pisot root $\beta \approx 3.196$ and associated complex conjugates $\beta_2, \bar{\beta}_2 \approx -0.098 \pm 0.785i$ with norm less than 1.

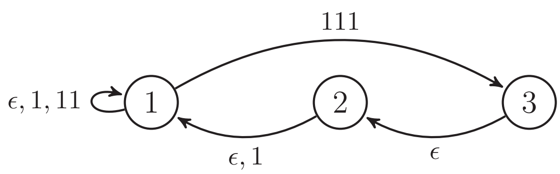


FIGURE 2.10. Prefix graph of σ.

We want to determine the representation space for the substitution σ. Setting $K = \mathbb{Q}(\beta)$, we know that the Archimedean part of the representation space is $\mathbb{C}$, while for the non-Archimedean part we have to compute the prime ideal factorization of $2\mathcal{O}$:

$$(2) = (2,\beta)^2\,(2,\beta-1) = \underbrace{(\beta)^2}_{\mathfrak{p}_1^2}\,\underbrace{(-1-\beta^2)}_{\mathfrak{p}_2}.$$

We have $|\beta|_{\mathfrak{p}_1} = \frac{1}{2}$, $|\beta|_{\mathfrak{p}_2} = 1$, hence the non-Archimedean completion we have to consider is $K_{\mathfrak{p}_1}$, which is an extension of degree $e_1 f_1 = 2$ of $\mathbb{Q}_2$.

There exist only 7 non-isomorphic quadratic extensions of $\mathbb{Q}_2$, and one can check that $K_{\mathfrak{p}_1} \cong \mathbb{Q}_2(\sqrt{7})$. Furthermore β is a uniformiser in $K_{\mathfrak{p}_1}$, thus we can express every element of this completion as $\sum_{i=m}^{\infty} d_i\beta^i$, with $d_i \in \{0,1\}$ and some $m \in \mathbb{Z}$. The canonical embedding is

$$\delta_c : \mathbb{Q}(\beta) \to \mathbb{C} \times \mathbb{Q}_2(\sqrt{7}), \quad a_0 + a_1\beta + a_2\beta^2 \mapsto \left(a_0 + a_1\beta_2 + a_2\beta_2^2, \sum_{i=m}^{\infty} d_i\beta^i\right).$$

We represent each element of $\mathbb{Q}_2(\sqrt{7})$ with the Monna map $\mathbb{Q}_2(\sqrt{7}) \to \mathbb{R}^+$, $\sum_{i=m}^{\infty} d_i\beta^i \mapsto \sum_{i=m}^{\infty} d_i 2^{-i-1}$ described in Section 1.3.4. In Figure 2.11 the tiles associated with the substitution are represented. We choose $\mathbf{v}_\beta = (\beta^2/2, \beta, 1)$ as left eigenvector of M_σ. With this choice we have the set of digits $\mathcal{D} = \{0, \beta^2/2, \beta^2, 3\beta^2/2\}$ for the Dumont-Thomas expansions. Furthermore we have that the $\mathfrak{p}$-adic part of the central tile is contained in $\mathbb{Z}_2[\sqrt{7}]$.

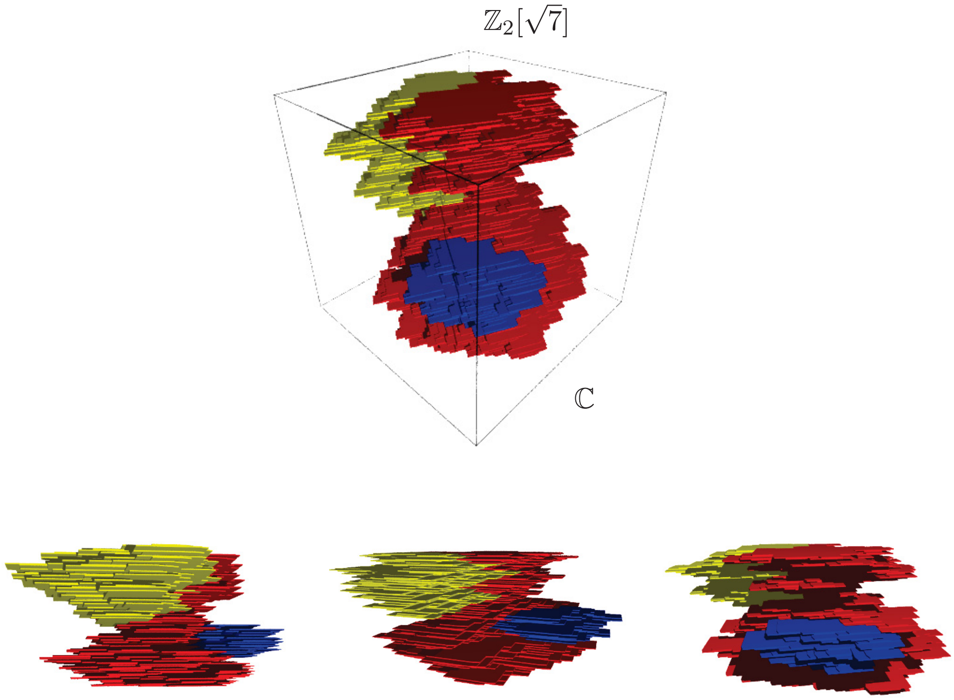


FIGURE 2.11. Pictures of the central tile $\mathcal{R}_\sigma$ divided in the red (gray) subtile $\mathcal{R}_\sigma(1)$, the blue (dark gray) $\mathcal{R}_\sigma(2)$ and the yellow (light gray) $\mathcal{R}_\sigma(3)$.

CHAPTER 3

Tilings for Pisot beta-numeration

For a (non-unit) Pisot number β, several collections of tiles are associated with β-numeration. This includes an aperiodic and a periodic one made of Rauzy fractals, a periodic one induced by the natural extension of the β-transformation and a Euclidean one made of integral beta-tiles. We show that all these collections (except possibly the periodic translation of the central tile) are tilings if one of them is a tiling or, equivalently, the weak finiteness property (W) or a spectral condition on the boundary graph hold. We also obtain new results on rational numbers with purely periodic β-expansions; in particular, we calculate $\gamma(\beta)$ for all quadratic β with $\beta^2 = a\beta + b$, $\gcd(a, b) = 1$. This chapter is based on [MS14]. Background notions on beta-numeration can be found in Section 1.2.2.

3.1. Tiles

Beta-tiles. For $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$, define the *x-tile* (or *Rauzy fractal*) as

$$\mathcal{R}(x) = \lim_{k\to\infty} \delta_c\big(\beta^k\, T_\beta^{-k}(x)\big) \subseteq \mathbb{K}_\beta^c, \tag{3.1}$$

where the limit is taken with respect to the Hausdorff distance, and let

$$\mathcal{C}_{\mathrm{aper}} = \big\{\mathcal{R}(x) : x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)\big\},$$
$$\tilde{\mathcal{C}}_{\mathrm{aper}} = \big\{\mathcal{R}(x) : x \in \mathbb{Z}[\beta] \cap [0, 1)\big\} \subseteq \mathcal{C}_{\mathrm{aper}}.$$

Note that the limit in (3.1) exists since $\beta^k\, T_\beta^{-k}(x) \subseteq \beta^{k+1}\, T_\beta^{-k-1}(x)$ for all $k \in \mathbb{N}$. Recall that

$$L = \big\langle \widehat{V} - \widehat{V}\big\rangle_{\mathbb{Z}} \subseteq \mathbb{Z}[\beta]$$

is the $\mathbb{Z}$-module generated by the differences of elements in $\widehat{V}$ and let

$$\mathcal{C}_{\mathrm{per}} = \big\{\delta_c(x) + \mathcal{R}(0) : x \in L\big\}.$$

The periodic collection of tiles $\mathcal{C}_{\mathrm{per}}$ is locally finite only when

$$\mathrm{rank}(L) = \deg(\beta) - 1 \tag{3.2}$$

holds, which is (QM), an analogue of the *quotient mapping condition* defined in [ST09]. A sufficient condition for (QM) is that $\#V = \deg(\beta)$. In Section 3.6.4, we give examples with $\#V > \deg(\beta)$ where (QM) holds and does not hold, respectively.

Integral beta-tiles. For $x \in \mathbb{Z}[\beta] \cap [0, 1)$, the *integral x-tile*

$$\mathcal{S}(x) = \lim_{k\to\infty} \delta_\infty^c\Big(\beta^k\big(T_\beta^{-k}(x) \cap \mathbb{Z}[\beta]\big)\Big) \subseteq \mathbb{K}_\infty^c \tag{3.3}$$

was introduced in [BSS+11] in the context of SRS tiles; see also [ST13]. Let

$$\mathcal{C}_{\mathrm{int}} = \big\{\mathcal{S}(x) : x \in \mathbb{Z}[\beta] \cap [0, 1)\big\}.$$

If β is an algebraic unit, then $\mathbb{Z}[\beta] = \mathbb{Z}[\beta^{-1}]$ and $\mathcal{S}(x) = \mathcal{R}(x)$, $\mathcal{C}_{\mathrm{int}} = \tilde{\mathcal{C}}_{\mathrm{aper}} = \mathcal{C}_{\mathrm{aper}}$.

Natural extension. Recall that if $(Y, \mathcal{B}_Y, \nu, S)$ is a non-invertible measure-preserving dynamical system, an invertible measure-preserving dynamical system $(X, \mathcal{B}_X, \mu, T)$ is called a *natural extension* of $(Y, \mathcal{B}_Y, \nu, S)$ if $(Y, \mathcal{B}_Y, \nu, S)$ is a factor of $(X, \mathcal{B}_X, \mu, T)$ and the factor map ϕ satisfies $\bigvee_{m=0}^{\infty} T^m \phi^{-1} \mathcal{B}_Y = \mathcal{B}_X$, where $\bigvee_{m=0}^{\infty} T^m \phi^{-1} \mathcal{B}_Y$ is the smallest σ-algebra containing the σ-algebras $T^m \phi^{-1} \mathcal{B}_Y$ for all $m \in \mathbb{N}$.

We give a version of the natural extension of the β-transformation T_β with nice algebraic and geometric properties, in the case where β is a Pisot number, not necessarily unit. We will do this using the x-tiles defined above. Let

$$\mathscr{X} = \bigcup_{v \in V} \Big([v, \widehat{v}) \times \big(\delta_c(v) - \mathcal{R}(v)\big) \Big) \subseteq \mathbb{K}_\beta,$$
$$\mathscr{T} : \mathscr{X} \to \mathscr{X}, \quad \mathbf{z} \mapsto \beta\, \mathbf{z} - \delta(\lfloor \beta\, \pi_1(\mathbf{z}) \rfloor),$$

$$\mathcal{C}_{\mathrm{ext}} = \big\{ \delta(x) + \overline{\mathscr{X}} : x \in \mathbb{Z}[\beta^{-1}] \big\},$$
$$\tilde{\mathcal{C}}_{\mathrm{ext}} = \big\{ \delta(x) + \overline{\mathscr{X}} : x \in \mathbb{Z}[\beta] \big\}.$$

The set $\mathscr{X}$ is the domain and $\mathscr{T}$ the transformation of our natural extension of the beta-transformation on $[0, 1)$. Note that one usually requires the natural extension domain to be compact. Here, we often prefer working with $\mathscr{X}$ instead of its closure because it has some nice properties, e.g., it characterises the purely periodic expansions.

Boundary graph. The nodes of the *boundary graph* are the triples $[v, x, w] \in V \times \mathbb{Z}[\beta] \times V$ such that $x \neq 0$, $\delta_c(x) \in \mathcal{R}(v) - \mathcal{R}(w) + \delta_c(w - v)$, and $w - \widehat{v} < x < \widehat{w} - v$. There is an edge

$$[v, x, w] \xrightarrow{(a,b)} [v_1, x_1, w_1] \;\Leftrightarrow\; a, b \in \mathcal{D},\ x_1 = \tfrac{b-a+x}{\beta},\ \tfrac{a+v}{\beta} \in [v_1, \widehat{v_1}),\ \tfrac{b+w}{\beta} \in [w_1, \widehat{w_1}).$$

This graph provides expansions of the points that lie in two different elements of $\mathcal{C}_{\mathrm{aper}}$. If $\mathcal{C}_{\mathrm{aper}}$ forms a tiling, then these points are exactly the boundary points of the tiles.

In [ABBS08], a slightly different boundary graph is defined that determines the boundary of subtiles instead of that of Rauzy fractals. In their definition, x may be in $\mathbb{Z}[\beta^{-1}]$ and it is shown to be in $\mathcal{O}$. We will see that $\mathbb{Z}[\beta]$ is sufficient.

Purely periodic expansions. Let

$$\mathrm{Pur}(\beta) = \big\{ x \in [0, 1) : T_\beta^k(x) = x \text{ for some } k \geq 1 \big\}.$$

be the set of numbers with purely periodic β-expansion. By [Ber77, Sch80], we know that

$$\exists\, k \geq 0 :\ T_\beta^k(x) \in \mathrm{Pur}(\beta) \quad \text{if and only if} \quad x \in \mathbb{Q}(\beta) \cap [0, 1). \tag{3.4}$$

Furthermore, the set $\mathrm{Pur}(\beta)$ was characterised in [HI97, IR05, BS07] by

$$x \in \mathrm{Pur}(\beta) \quad \text{if and only if} \quad x \in \mathbb{Q}(\beta),\ \delta(x) \in \mathscr{X}; \tag{3.5}$$

see also [KS12]. In particular, we have

$$\mathbb{Q} \cap \mathrm{Pur}(\beta) \subseteq \mathbb{Z}_{N(\beta)} = \big\{ p/q \in \mathbb{Q} : \gcd(q, N(\beta)) = 1 \big\};$$

see e.g. [ABBS08, Lemma 4.1]. Here, $N(\beta)$ denotes the norm of the algebraic number β. We study the quantity

$$\gamma(\beta) = \sup\big\{r \in [0,1] : \mathbb{Z}_{N(\beta)} \cap [0,r) \subseteq \mathrm{Pur}(\beta)\big\} \tag{3.6}$$

that was introduced in [Aki98].

Weak finiteness. The arithmetical property

$$\forall\, x \in \mathrm{Pur}(\beta) \cap \mathbb{Z}[\beta]\ \exists\, y \in [0, 1-x),\ n \in \mathbb{N} :\ T_\beta^n(x+y) = T_\beta^n(y) = 0, \tag{W}$$

turns out to be equivalent to the tiling property of our collections.

3.2. Main results

In the following theorem, we list some important properties of the x-tiles. Most of them can be proved exactly as in the unit case, see e.g. [KS12]. Some of them can also be found in [BS07, ABBS08] or are direct consequences of the more general results proven in [MT14] in the substitution settings. For convenience, we provide a full proof in Section 3.4.

THEOREM 3.1. *Let β be a Pisot number. For each $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$, the following hold:*

(i) $\mathcal{R}(x)$ is a non-empty compact set that is the closure of its interior.

(ii) The boundary of $\mathcal{R}(x)$ has Haar measure zero.

(iii) $\mathcal{R}(x) = \bigcup_{y \in T_\beta^{-1}(x)} \beta\, \mathcal{R}(y)$, and the union is disjoint in Haar measure.

(iv) $\mathcal{R}(x) - \delta_c(x) \subseteq \mathcal{R}(v) - \delta_c(v)$ for all $v \in V$ with $v \le x$.

(v) $\mathcal{R}(x) - \delta_c(x) \supseteq \mathcal{R}(v) - \delta_c(v)$ for all $v \in V$ with $\widehat{v} > x$.

Moreover, we have

$$\delta\big(\mathbb{Z}[\beta^{-1}]\big) + \mathscr{X} = \mathbb{K}_\beta, \qquad \delta\big(\mathbb{Z}[\beta]\big) + \mathscr{X} = Z, \tag{3.7}$$

$$\bigcup_{x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)} \mathcal{R}(x) = \mathbb{K}_\beta^c, \qquad \bigcup_{x \in \mathbb{Z}[\beta] \cap [0,1)} \mathcal{R}(x) = Z^c,$$

and

$$\overline{\mathscr{X}} = \overline{\bigcup_{x \in \mathbb{Z}[\beta] \cap [0,1)} \delta(x) - \{0\} \times \mathcal{R}(x)}. \tag{3.8}$$

The following theorem is informally stated in [ABBS08] and other papers; see [KS12] for the unit case. Here, B and $\mathscr{B}$ denote the Borel σ-algebras on $X = [0,1)$ and $\mathscr{X}$, respectively. The set $\mathscr{X}$ is equipped with the Haar measure μ, while X is equipped with the measure $\mu \circ \pi_1^{-1}$, which is an absolutely continuous invariant measure for T.

THEOREM 3.2. *Let β be a Pisot number. The dynamical system $(\mathscr{X}, \mathscr{B}, \mu, \mathscr{T})$ is a natural extension of $([0,1), B, \mu \circ \pi_1^{-1}, T)$.*

Some of the following properties of integral β-tiles can be found in [BSS+11]; the main novelty is that we can show that the boundary has measure zero.

THEOREM 3.3. *Let β be a Pisot number. For each $x \in \mathbb{Z}[\beta] \cap [0,1)$, the following hold:*

(i) $\mathcal{S}(x)$ is a non-empty compact set.

(ii) *The boundary of* $\mathcal{S}(x)$ *has Lebesgue measure zero.*

(iii) $\mathcal{S}(x) = \bigcup_{y \in T_\beta^{-1}(x) \cap \mathbb{Z}[\beta]} \beta\, \mathcal{S}(y)$.

(iv) *For all* $y \in (x + \beta^k \mathbb{Z}[\beta]) \cap [v, \widehat{v})$, $k \in \mathbb{N}$,

$$d_H\big(\mathcal{S}(x) - \delta_\infty^c(x), \mathcal{S}(y) - \delta_\infty^c(y)\big) \le 2 \operatorname{diam} \pi_\infty^c(\beta^k \mathcal{R}(0)),$$

where $v \in V$ *is chosen such that* $x \in [v, \widehat{v})$, *and* d_H *denotes the Hausdorff distance with respect to some metric on* $\mathbb{K}_\infty^c$.

(v) *If* β *is quadratic, then* $\mathcal{S}(x)$ *is an interval that intersects* $\bigcup_{y \in \mathbb{Z}[\beta] \cap [0,1) \setminus \{x\}} \mathcal{S}(y)$ *only at its endpoints.*

Moreover, we have

$$\bigcup_{x \in \mathbb{Z}[\beta] \cap [0,1)} \mathcal{S}(x) = \mathbb{K}_\infty^c, \tag{3.9}$$

and, if $\deg(\beta) \ge 2$,

$$\mathcal{R}(v) = \overline{\bigcup_{x \in \mathbb{Z}[\beta] \cap [v, \widehat{v})} \delta_c(v - x) + \mathcal{S}(x) \times \delta_f(\{0\})} \quad \text{for all} \quad v \in V, \tag{3.10}$$

$$\overline{\mathscr{X}} = \overline{\bigcup_{x \in \mathbb{Z}[\beta] \cap [0,1)} \delta(x) - \{0\} \times \mathcal{S}(x) \times \delta_f(\{0\})}. \tag{3.11}$$

A series of equivalent tiling conditions constitutes the core of this paper.

Theorem 3.4. *Let* β *be a Pisot number. Then the collections* $\mathcal{C}_{\text{ext}}$, $\tilde{\mathcal{C}}_{\text{ext}}$, $\mathcal{C}_{\text{aper}}$, $\tilde{\mathcal{C}}_{\text{aper}}$, *and* $\mathcal{C}_{\text{int}}$ *are multiple tilings of* $\mathbb{K}_\beta$, Z, $\mathbb{K}_\beta^c$, Z^c, *and* $\mathbb{K}_\infty^c$, *respectively, and they all have the same covering degree. The following statements are equivalent:*

(i) *All collections* $\mathcal{C}_{\text{ext}}$, $\tilde{\mathcal{C}}_{\text{ext}}$, $\mathcal{C}_{\text{aper}}$, $\tilde{\mathcal{C}}_{\text{aper}}$, *and* $\mathcal{C}_{\text{int}}$ *are tilings.*

(ii) *One of the collections* $\mathcal{C}_{\text{ext}}$, $\tilde{\mathcal{C}}_{\text{ext}}$, $\mathcal{C}_{\text{aper}}$, $\tilde{\mathcal{C}}_{\text{aper}}$, *and* $\mathcal{C}_{\text{int}}$ *is a tiling.*

(iii) *One of the collections* $\mathcal{C}_{\text{ext}}$, $\tilde{\mathcal{C}}_{\text{ext}}$, $\mathcal{C}_{\text{aper}}$, $\tilde{\mathcal{C}}_{\text{aper}}$, *and* $\mathcal{C}_{\text{int}}$ *has an exclusive point.*

(iv) *Property* (W) *holds.*

(v) *The spectral radius of the boundary graph is less than* β.

If (QM) *holds, then the following statement is also equivalent to the ones above:*

(vi) $\mathcal{C}_{\text{per}}$ *is a tiling of* Z^c.

By Theorem 3.3 (v) or e.g. by [ARS04], the equivalent statements of the theorem hold when β is quadratic.

The following bound and formula for $\gamma(\beta)$ (defined in (3.6)) simplify those that can be found in [ABBS08].

Theorem 3.5. *Let* β *be a Pisot number. Then*

$$\gamma(\beta) \ge \inf\left(\{1\} \cup \bigcup_{v \in V} \big\{x \in \mathbb{Q} \cap [v, \widehat{v}) : \delta_\infty^c(v - x) \in \pi_\infty^c\big(Z^c \setminus \mathcal{R}(v)\big)\big\}\right). \tag{3.12}$$

If moreover $\overline{\delta_f(\mathbb{Q})} = \mathbb{K}_f$, *then equality holds in* (3.12).

Note that $\overline{\delta_\infty^c(\mathbb{Q})}$ is a line in $\mathbb{K}_\infty^c$, thus we essentially have to determine the intersection of a line with the projection of the complement of $\mathcal{R}(v)$. We are able to calculate the explicit value for $\gamma(\beta)$ for a large class of quadratic Pisot numbers.

THEOREM 3.6. *Let β be a quadratic Pisot number with $\beta^2 = a\beta+b$, $a \ge b \ge 1$. Then*

$$\gamma(\beta) \ge \max\left\{0, 1 - \frac{(b-1)b\,\beta}{\beta^2-b^2}\right\}, \tag{3.13}$$

and equality holds if $\gcd(a,b) = 1$. *We have* $\frac{(b-1)b\,\beta}{\beta^2-b^2} < 1$ *if and only if* $(b-1)b < a$.

For $a = b = 2$, numerical experiments suggest that $\gamma(\beta) \approx 0.9148$, while (3.13) only gives $\gamma(\beta) \ge 0$. Thus we believe that equality may not hold in (3.13) if $\gcd(a,b) > 1$.

3.3. An example

We illustrate our different tilings for the example $\beta = 1+\sqrt{3}$, with $\beta^2 = 2\beta+2$. Here, the prime 2 ramifies in $\mathcal{O}$, thus we get the representation space $\mathbb{K}_\beta^c = \mathbb{R}\times\mathbb{K}_\mathrm{f}$, with $\mathbb{K}_\mathrm{f} \cong \mathbb{Q}_2^2$. Each element of $\mathbb{K}_\mathrm{f}$ can be written as $\sum_{j=k}^\infty \delta_\mathrm{f}(d_j\beta^j)$, with $d_j \in \{0,1\}$, and we represent it by $\sum_{j=k}^\infty d_j 2^{-j-1}$ in our pictures (see Section 1.3.4).

In Figure 3.1, a patch of the aperiodic tiling $\mathcal{C}_\mathrm{aper}$ together with the corresponding integral beta-tiles (that form $\mathcal{C}_\mathrm{int}$) is represented. The aperiodic tiling $\widetilde{\mathcal{C}}_\mathrm{aper}$ constitutes the "lowest stripe" of $\mathcal{C}_\mathrm{aper}$. Another possibility to tile the stripe Z^c is given by the periodic tiling $\mathcal{C}_\mathrm{per}$ that is sketched in Figure 3.2. In Figure 3.3, the natural extension domain is shown, which tiles $\mathbb{K}_\beta$ and Z periodically; see Figure 3.4 and 3.5.

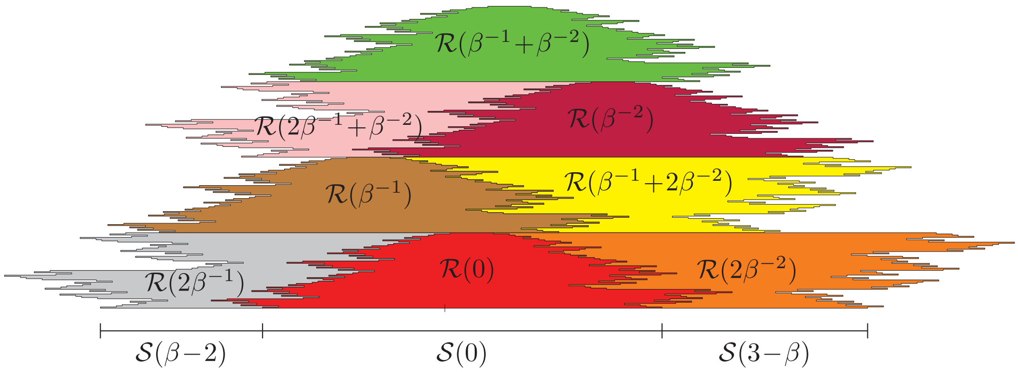


FIGURE 3.1. The patch $\beta^{-2}\mathcal{R}(0)$ of the aperiodic tiling $\mathcal{C}_\mathrm{aper}$ and the corresponding integral beta-tiles, $\beta^2 = 2\beta+2$.

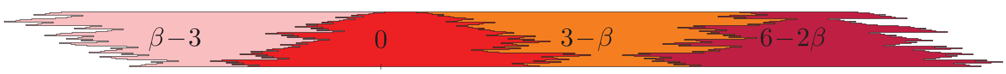


FIGURE 3.2. A patch of $\mathcal{C}_\mathrm{per} = \{\delta_c(x) + \mathcal{R}(0) : x \in \mathbb{Z}(\beta-3)\}$, $\beta^2 = 2\beta+2$.

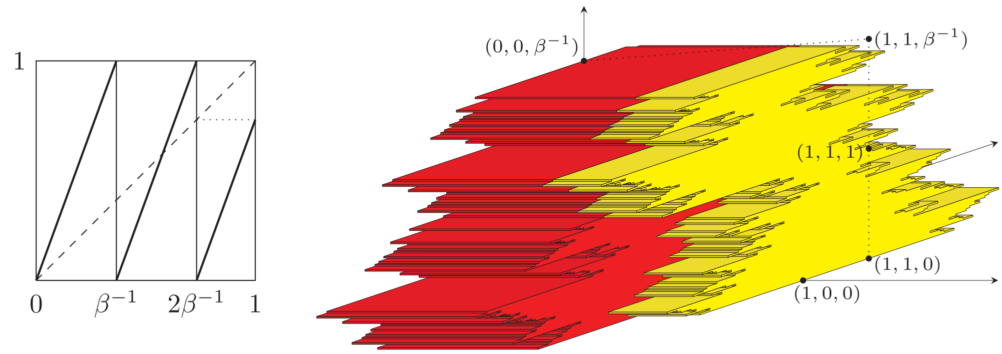


FIGURE 3.3. β-transformation and natural extension domain $\mathscr{X}$ for $\beta^2 = 2\beta + 2$.

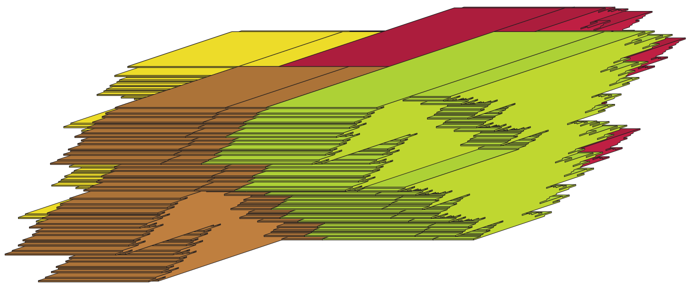


FIGURE 3.4. Periodic tiling $\tilde{\mathcal{C}}_{\mathrm{ext}} \subseteq \mathcal{C}_{\mathrm{ext}}$, $\beta^2 = 2\beta + 2$. The following tiles are represented: $\mathscr{X}$ (yellow), $\mathscr{X} + \delta(1)$ (purple), $\mathscr{X} + \delta(\beta - 2)$ (brown), $\mathscr{X} + \delta(\beta - 1)$ (light green).

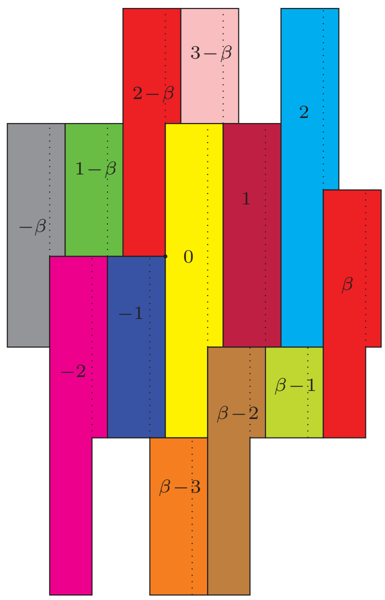


FIGURE 3.5. The intersection of $\tilde{\mathcal{C}}_{\mathrm{ext}}$ with $\mathbb{K}_\infty \times \delta_{\mathrm{f}}(\{0\})$.

3.4. Properties of Rauzy fractals and the natural extension

Proof of Theorem 3.1. Let $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$. Each element of $\mathcal{R}(x)$ is the limit of elements of $\delta_c(\beta^k\, T_\beta^{-k}(x))$ and hence of the form

$$\mathbf{z} = \lim_{k\to\infty} \delta_c\bigg(\sum_{j=0}^{k-1} a_j\beta^j + x\bigg) = \delta_c(x) + \sum_{j=0}^{\infty} \delta_c(a_j\beta^j),$$

with $a_j \in \mathcal{D} = \{0,1,\dots,\lceil\beta\rceil - 1\}$ and $\sum_{j=0}^{k-1} a_j\beta^j + x \in [0,\beta^k)$ for all $k \in \mathbb{N}$. Thus our definition of the x-tiles is essentially the same as in the other papers on this topic. In particular, $\mathcal{R}(x)$ is a compact set with uniformly bounded diameter.

The set equation in Theorem 3.1 (iii) is a direct consequence of our definition.

If $0 \le y \le x$, then we have

$$\beta^k\, T_\beta^{-k}(x) - x \subseteq \beta^k\, T_\beta^{-k}(y) - y$$

for all $k \in \mathbb{N}$, which proves Theorem 3.1 (iv). Let now $v \in V$ with $\widehat{v} > x$. To see that

$$\beta^k\, T_\beta^{-k}(x) - x \supseteq \beta^k\, T_\beta^{-k}(v) - v$$

for all $k \in \mathbb{N}$, suppose that there exists $z \in (\beta^k\, T_\beta^{-k}(v) - v) \setminus (\beta^k\, T_\beta^{-k}(x) - x)$, and assume that k is minimal such that this set is non-empty. Then we have $z + v < \beta^k \le z + x$. Since $T_\beta^j(\frac{z+v}{\beta^k}) \in T_\beta^{j-k}(v)$ for $1 \le j \le k$, the minimality of k gives that $\beta^{k-j}T_\beta^j(\frac{z+v}{\beta^k}) + x - v \in \beta^{k-j}\, T_\beta^{j-k}(x)$, thus $\beta^{k-j}T_\beta^j(\frac{z+v}{\beta^k}) + \beta^k - z - v < \beta^{k-j}$, which implies that $T_\beta^j(1^-) = \beta^j + T_\beta^j(\frac{z+v}{\beta^k}) - \frac{z+v}{\beta^{k-j}}$. Hence $T_\beta^k(1^-) = \beta^k - z$ and thus $v < T_\beta^k(1^-) \le x$, which contradicts the assumption that $\widehat{v} > x$. This proves Theorem 3.1 (v). In particular, we have that

$$\mathcal{R}(x) - \delta_c(x) = \mathcal{R}(v) - \delta_c(v) \quad \text{for all} \quad x \in \mathbb{Z}[\beta^{-1}] \cap [v,\widehat{v}),\ v \in V. \tag{3.14}$$

We now consider the covering properties of our collections of tiles, following mainly [KS12]. We start with a short proof of (3.5). Let $x \in [0,1)$ with $T_\beta^k(x) = x$. Then $x \in \mathbb{Q}(\beta)$ and

$$\delta_c(0) = \lim_{n\to\infty} \delta_c(\beta^{nk}x) \in \lim_{n\to\infty} \delta_c(\beta^{nk}T_\beta^{-nk}(x)) \in \mathcal{R}(x),$$

where we have extended the definition of $\mathcal{R}(x)$ to $x \in \mathbb{Q}(\beta)$. If $v \in V$ is such that $x \in [v,\widehat{v})$, then (3.14) gives that

$$\delta(x) \in \{x\} \times (\delta_c(x) - \mathcal{R}(x)) = \{x\} \times (\delta_c(v) - \mathcal{R}(v)) \subseteq \mathscr{X}.$$

On the other hand, let $x \in \mathbb{Q}(\beta)$ be such that $\delta(x) \in \mathscr{X}$. If $q \in \mathbb{Z}$ is such that $x \in \frac{1}{q}\mathbb{Z}[\beta^{-1}]$, then we have $\mathscr{T}^{-n}(\delta(x)) \subseteq \delta(\frac{1}{q}\mathbb{Z}[\beta^{-1}])$ for all $n \in \mathbb{N}$. Since $\delta(\frac{1}{q}\mathbb{Z}[\beta^{-1}])$ is a lattice in $\mathbb{K}_\beta$ by Lemma 1.23 and $\mathscr{X}$ is bounded, the set $\bigcup_{n\in\mathbb{N}} \mathscr{T}^{-n}(\delta(x))$ is finite, thus $\mathscr{T}^k(\delta(z)) = \delta(z)$ for some $k \ge 1$, $z \in \mathbb{Q}(\beta) \cap [0,1)$, with $\delta(z) \in \mathscr{T}^{-n}(\delta(x))$, $n > k$. Since $\mathscr{T}(\delta(z)) = \delta(T_\beta(z))$, we obtain that $T_\beta^k(z) = z$ and $T_\beta^n(z) = x$, thus $T_\beta^k(x) = x$.

Now, we use (3.5) to prove that $\delta(\mathbb{Z}[\beta^{-1}]) + \mathscr{X} = \mathbb{K}_\beta$. We first show that

$$\delta(\mathbb{Q}(\beta)) \subseteq \delta(\mathbb{Z}[\beta^{-1}]) + \mathscr{X}. \tag{3.15}$$

Let $x \in \mathbb{Q}(\beta)$. Then the sequence $(\beta^k x \bmod \mathbb{Z}[\beta^{-1}])_{k\in\mathbb{Z}}$ is periodic; indeed, choosing $q \in \mathbb{Z}$ such that $x \in \frac{1}{q}\mathbb{Z}[\beta^{-1}]$, we have $\beta^k x \in \frac{1}{q}\mathbb{Z}[\beta^{-1}]$ for all $k \in \mathbb{Z}$, and the periodicity of $(\beta^k x \bmod \mathbb{Z}[\beta^{-1}])_{k\in\mathbb{Z}}$ follows from the finiteness of $\frac{1}{q}\mathbb{Z}[\beta^{-1}]/\mathbb{Z}[\beta^{-1}]$. By (3.4) and (3.5), we have $\delta(T_\beta^k(x - \lfloor x\rfloor)) \in \mathscr{X}$ for all sufficiently large $k \in \mathbb{N}$. Thus we can choose $k \in \mathbb{N}$ such that $\beta^k x \equiv x \bmod \mathbb{Z}[\beta^{-1}]$ and $\delta(T_\beta^k(x - \lfloor x\rfloor)) \in \mathscr{X}$. As $T_\beta^k(x - \lfloor x\rfloor) \equiv \beta^k x \bmod \mathbb{Z}[\beta^{-1}]$, we obtain that $\delta(x) \in \mathscr{X} + \delta(\mathbb{Z}[\beta^{-1}])$, i.e., (3.15) holds.

Since $\overline{\mathscr{X}}$ is compact and $\delta(\mathbb{Z}[\beta^{-1}])$ is a lattice, (3.15) implies that

$$\delta(\mathbb{Z}[\beta^{-1}]) + \overline{\mathscr{X}} = \overline{\delta(\mathbb{Q}(\beta))} = \mathbb{K}_\beta.$$

Observe that $\mathscr{X}$ differs only slightly from its closure:

$$\overline{\mathscr{X}} \setminus \mathscr{X} = \bigcup_{v\in V} \Big(\{\widehat{v}\} \times \big(\delta_c(v) - \mathcal{R}(v)\big) \setminus \big(\delta_c(\widehat{v}) - \mathcal{R}(\widehat{v})\big)\Big), \tag{3.16}$$

where $\mathcal{R}(1) = \emptyset$. As $\mathscr{X}$ is a finite union of products of a left-closed interval with a compact set, the complement of $\delta(\mathbb{Z}[\beta^{-1}]) + \mathscr{X}$ in $\mathbb{K}_\beta$ is either empty or has positive measure. Since $\mu(\overline{\mathscr{X}} \setminus \mathscr{X}) = 0$, we obtain that

$$\delta(\mathbb{Z}[\beta^{-1}]) + \mathscr{X} = \mathbb{K}_\beta. \tag{3.17}$$

Since $\mathcal{R}(v) - \delta(v) \subseteq \overline{\delta_c(\mathbb{Z}[\beta])}$ for all $v \in V$, we have $\mathscr{X} \subseteq Z$. By Lemma 1.29, we have thus $(\delta(x) + \mathscr{X}) \cap Z = \emptyset$ for all $x \in \mathbb{Z}[\beta^{-1}] \setminus \mathbb{Z}[\beta]$. Together with (3.17), this implies that

$$\delta(\mathbb{Z}[\beta]) + \mathscr{X} = Z.$$

For each $x \in \mathbb{Z}[\beta^{-1}] \cap [v, \widehat{v})$, $v \in V$, we have

$$\begin{aligned} \mathscr{X} \cap \{x\} \times \mathbb{K}_\beta^c &= \{x\} \times \big(\delta_c(v) - \mathcal{R}(v)\big) \\ &= \{x\} \times \big(\delta_c(x) - \mathcal{R}(x)\big) = \delta(x) - \{0\} \times \mathcal{R}(x), \end{aligned} \tag{3.18}$$

Since $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$, we obtain (3.8). Rewriting (3.18), we get $\big(\delta(x) - \mathscr{X}\big) \cap \{0\} \times \mathbb{K}_\beta^c = \{0\} \times \mathcal{R}(x)$, which shows together with (3.17) that

$$\bigcup_{x\in\mathbb{Z}[\beta^{-1}]\cap[0,1)} \mathcal{R}(x) = \mathbb{K}_\beta^c. \tag{3.19}$$

Since $\mathcal{R}(x) \cap Z^c = \emptyset$ for all $x \in \mathbb{Z}[\beta^{-1}] \setminus \mathbb{Z}[\beta]$ by Lemma 1.29, we have

$$\bigcup_{x\in\mathbb{Z}[\beta]\cap[0,1)} \mathcal{R}(x) = Z^c. \tag{3.20}$$

By (3.19) and Baire's theorem, $\mathcal{R}(x)$ has non-empty interior for some $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$. Using the set equations in Theorem 3.1 (iii) and (3.14), we obtain that $\mathcal{R}(x)$ has non-empty interior for all $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$. Consequently, the set equations also imply that $\mathcal{R}(x)$ is the closure of its interior; see e.g. [KS12] or Chapter 2 for more details. This proves Theorem 3.1 (i).

For the proof of Theorem 3.1 (ii), we follow again [KS12] and prove first that $\mathscr{T}$ is bijective up to a set of measure zero. First note that $\mathscr{T}(\mathscr{X}) = \mathscr{X}$. Partitioning $\mathscr{X}$ into the sets

$$\mathscr{X}_a = \{\mathbf{z} \in \mathscr{X} : \lfloor\beta\, \pi_1(\mathbf{z})\rfloor = a\} \qquad (a \in \mathcal{D}),$$

we have $\mathscr{T}(\mathbf{z}) = \beta\,\mathbf{z} - \delta(a)$ for all $\mathbf{z} \in \mathscr{X}_a$. Thus $\mathscr{T}$ is injective on each $\mathscr{X}_a$, $a \in \mathcal{D}$, and

$$\sum_{a\in\mathcal{D}} \mu\big(\mathscr{T}(\mathscr{X}_a)\big) = \sum_{a\in\mathcal{D}} \mu(\beta\,\mathscr{X}_a) = \sum_{a\in\mathcal{D}} \mu(\mathscr{X}_a)$$
$$= \mu(\mathscr{X}) = \mu\big(\mathscr{T}(\mathscr{X})\big) = \mu\Big(\bigcup_{a\in\mathcal{D}} \mathscr{T}(\mathscr{X}_a)\Big),$$

where the second equality holds by the product formula $\prod_{\mathfrak{p}\in S} |\beta|_{\mathfrak{p}} = 1$. Hence $\mu(\mathscr{T}(\mathscr{X}_a) \cap \mathscr{T}(\mathscr{X}_b)) = 0$ for all $a, b \in \mathcal{D}$ with $a \neq b$, and $\mathscr{T}$ is bijective up to a set of measure zero.

For each $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$ and sufficiently small $\varepsilon > 0$, we have

$$\mathscr{T}^{-1}\Big([x, x+\varepsilon) \times \big(\delta_c(x) - \mathcal{R}(x)\big)\Big) = \bigcup_{y\in T_\beta^{-1}(x)} [y, y+\varepsilon\beta^{-1}) \times \big(\delta_c(y) - \mathcal{R}(y)\big).$$

Since this union is disjoint, the union in Theorem 3.1 (iii) is disjoint in Haar measure. Thus if $\mathcal{R}(y)$, $y \in T_\beta^{-k}(x)$, is in the interior of $\beta^{-k}\,\mathcal{R}(x)$, its boundary has measure zero. As $\mathcal{R}(x)$ has non-empty interior and multiplication by β^{-1} is expanding on $\mathbb{K}_\beta^c$, we find for each $v \in V$ some $k \in \mathbb{N}$, $y \in T_\beta^{-k}(x) \cap [v, \widehat{v})$, such that $\mathcal{R}(y)$ is in the interior of $\beta^{-k}\,\mathcal{R}(x)$. Together with (3.14), this proves Theorem 3.1 (ii), which concludes the proof of Theorem 3.1.

Proof of Theorem 3.2. We have $\pi_1(\mathscr{X}) = [0, 1)$, $T_\beta \circ \pi_1 = \pi_1 \circ \mathscr{T}$, and we know from Section 3.4 that $\mathscr{T}$ is bijective on $\mathscr{X}$ up to a set of measure zero. It remains to show that

$$\bigvee_{k\in\mathbb{N}} \mathscr{T}^k \pi_1^{-1}(B) = \mathscr{B},$$

where $\bigvee_{k\in\mathbb{N}} \mathscr{T}^k \pi_1^{-1}(B)$ is the smallest σ-algebra containing the σ-algebras $\mathscr{T}^k$ $\pi_1^{-1}(B)$ for all $k \in \mathbb{N}$. It is clear that $\bigvee_{k\in\mathbb{N}} \mathscr{T}^k \pi_1^{-1}(B) \subseteq \mathscr{B}$. For the other inclusion, we show that we can always separate in $\bigvee_{k\in\mathbb{N}} \mathscr{T}^k \pi_1^{-1}(B)$ two points $\mathbf{z}, \tilde{\mathbf{z}} \in \mathscr{X}$ with $\mathbf{z} \neq \tilde{\mathbf{z}}$. If $\pi_1(\mathbf{z}) \neq \pi_1(\tilde{\mathbf{z}})$, then there are disjoint intervals $J, \tilde{J} \subseteq [0, 1)$ with $\mathbf{z} \in \pi_1^{-1}(J)$, $\tilde{\mathbf{z}} \in \pi_1^{-1}(\tilde{J})$. If $\pi_1(\mathbf{z}) = \pi_1(\tilde{\mathbf{z}})$, then consider the partition of $[0, 1)$ into continuity intervals of T_β^k for large k. For each continuity interval $J \subseteq [v, \widehat{v})$ of T_β^k, we have $\mathscr{T}^k \pi_1^{-1}(J) = T_\beta^k(J) \times (\delta_c(T_\beta^k(v)) - \beta^k\,\mathcal{R}(v))$. Since multiplication by β is contracting on $\mathbb{K}_\beta^c$, we can find $k \in \mathbb{N}$ such that $\mathbf{z} \in \mathscr{T}^k \pi_1^{-1}(J)$, $\tilde{\mathbf{z}} \in \mathscr{T}^k \pi_1^{-1}(\tilde{J})$, with two disjoint intervals $J, \tilde{J} \subseteq [0, 1)$.

3.5. Properties of integral beta-tiles

3.5.1. Basic properties. We first prove that, for each $x \in \mathbb{Z}[\beta] \cap [0, 1)$, $\mathcal{S}(x)$ is well defined and $\mathcal{S}(x) \neq \emptyset$. To this end, we show that $(\delta_\infty^c(\beta^k\,(T_\beta^{-k}(x) \cap \mathbb{Z}[\beta])))_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to the Hausdorff distance d_H. Since $\{0, 1, \ldots, |N(\beta)| - 1\}$ is a complete residue system of $\mathbb{Z}[\beta]/\beta\mathbb{Z}[\beta]$ and $|N(\beta)| \leq \beta$ because β is a Pisot number, we have $T_\beta^{-1}(y) \cap \mathbb{Z}[\beta] \neq \emptyset$ for all $y \in \mathbb{Z}[\beta] \cap [0, 1)$. Using that $T_\beta^{-k-1}(x) \cap \mathbb{Z}[\beta] = T_\beta^{-1}(T_\beta^{-k}(x) \cap \mathbb{Z}[\beta]) \cap \mathbb{Z}[\beta]$, we get

$$d_H\Big(\delta_\infty^c\big(\beta^{k+1}\,(T_\beta^{-k-1}(x) \cap \mathbb{Z}[\beta])\big), \delta_\infty^c\big(\beta^k\,(T_\beta^{-k}(x) \cap \mathbb{Z}[\beta])\big)\Big) \leq (\lceil\beta\rceil - 1)\,\|\delta_\infty^c(\beta^k)\|,$$

which tends to 0 exponentially fast as $k \to \infty$. This proves Theorem 3.3 (i).

Theorem 3.3 (iii) follows directly from the definition.

To show Theorem 3.3 (iv), let $x, y \in \mathbb{Z}[\beta] \cap [v, \widehat{v})$ with $x - y \in \beta^k \mathbb{Z}[\beta]$, $v \in V$. We know from the proof of Theorem 3.1 that $\beta^k T_\beta^{-k}(x) - x = \beta^k T_\beta^{-k}(y) - y$ and thus

$$\begin{aligned}\beta^k \big(T_\beta^{-k}(x) \cap \mathbb{Z}[\beta]\big) - x &= \beta^k \Big(\big(T_\beta^{-k}(y) + \beta^{-k}(x-y)\big) \cap \mathbb{Z}[\beta]\Big) - x \\ &= \beta^k \big(T_\beta^{-k}(y) \cap \mathbb{Z}[\beta]\big) - y.\end{aligned}$$

Therefore, we have

$$\begin{aligned}d_H\big(\mathcal{S}(x)-\delta_\infty^c(x), \mathcal{S}(y)-\delta_\infty^c(y)\big) &\le d_H\Big(\mathcal{S}(x)-\delta_\infty^c(x), \delta_\infty^c\big(\beta^k\big(T_\beta^{-k}(x)\cap\mathbb{Z}[\beta]\big)-x\big)\Big) \\ &\quad + d_H\Big(\delta_\infty^c\big(\beta^k\big(T_\beta^{-k}(y)\cap\mathbb{Z}[\beta]\big)-y\big), \mathcal{S}(y)-\delta_\infty^c(y)\Big) \\ &\le 2 \max_{z\in\mathbb{Z}[\beta]\cap[0,1)} d_H\Big(\mathcal{S}(z), \delta_\infty^c\big(\beta^k\big(T_\beta^{-k}(z)\cap\mathbb{Z}[\beta]\big)\big)\Big) \le 2\,\mathrm{diam}\,\pi_\infty^c\big(\beta^k\,\mathcal{R}(0)\big).\end{aligned}$$

3.5.2. Slices of Rauzy fractals and $\mathscr{X}$. An alternative definition of the integral x-tile, $x \in \mathbb{Z}[\beta] \cap [0, 1)$, could be

$$\mathcal{R}(x) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{0\}) = \mathcal{S}(x) \times \delta_{\mathrm{f}}(\{0\}). \tag{3.21}$$

Indeed, the inclusion $\supseteq$ follows from $\lim_{k\to\infty} \delta_{\mathrm{f}}(\beta^k \mathbb{Z}[\beta]) = \delta_{\mathrm{f}}(\{0\})$. For the other inclusion, let $\mathbf{z} \in \mathcal{R}(x) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{0\})$. By Theorem 3.1 (iii), there is a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in T_\beta^{-k}(x)$, $\mathbf{z} \in \beta^k \mathcal{R}(x_k)$ for all $k \in \mathbb{N}$. We have $x_k \in \mathbb{Z}[\beta]$, since otherwise we would have $\mathcal{R}(x_k) \cap Z^c = \emptyset$ and thus $\beta^k \mathcal{R}(x_k) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{0\}) = \emptyset$ by Lemma 1.29. Therefore, we have $\mathbf{z} \in \mathcal{S}(x) \times \delta_{\mathrm{f}}(\{0\})$, and (3.21) holds. Since the tiles can be obtained as the intersection with a "hyperplane", the equivalent of integral beta-tiles in [ST13] are called *intersective tiles*.

Now, the covering property (3.9) is a direct consequence of (3.20) and (3.21).

To prove (3.10), let $x \in \mathbb{Z}[\beta] \cap [v, \widehat{v})$, $v \in V$. Then we have

$$\begin{aligned}\mathcal{R}(v) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{v-x\}) &= \big(\mathcal{R}(x) + \delta_c(v-x)\big) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{v-x\}) \\ &= \delta_c(v-x) + \mathcal{R}(x) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{0\}) = \delta_c(v-x) + \mathcal{S}(x) \times \delta_{\mathrm{f}}(\{0\}).\end{aligned}$$

Since $\delta_{\mathrm{f}}(\mathbb{Z}[\beta] \cap (v - \widehat{v}, 0])$ is dense in $\overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$ if β is irrational and $\mathcal{R}(x)$ is the closure of its interior, we obtain (3.10).

Similarly to the x-tiles, we can express the natural extension domain by means of integral x-tiles. For each $x \in \mathbb{Z}[\beta] \cap [0, 1)$, we have

$$\begin{aligned}\mathscr{X} \cap \{x\} \times \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{x\}) &= \{x\} \times \Big(\big(\delta_c(x) - \mathcal{R}(x)\big) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{x\})\Big) \\ &= \delta(x) - \{0\} \times \big(\mathcal{R}(x) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{0\})\big) = \delta(x) - \{0\} \times \mathcal{S}(x) \times \delta_{\mathrm{f}}(\{0\}).\end{aligned}$$

By Lemma 1.28 and since $\mathbb{Z}[\beta]$ is a subgroup of finite index of $\mathcal{O}$, the set $\{(x, \delta_{\mathrm{f}}(x)) : x \in \mathbb{Z}[\beta] \cap [0, 1)\}$ is dense in $[0, 1] \times \overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$, provided that $\deg(\beta) \ge 2$. As $\mathscr{X}$ is the closure of its interior, we obtain (3.11).

3.5.3. Measure of the boundary. In the present subsection, we show that $\mu_\infty^c(\partial\mathcal{S}(x)) = 0$. The proof is similar to that of [ST13, Theorem 3 (i)].

Let $x \in \mathbb{Z}[\beta] \cap [0,1)$, and $X \subseteq \mathbb{K}_\infty^c$ be a rectangle containing $\mathcal{S}(x)$. Since $\beta^{-n}\,\partial\mathcal{S}(x) \subseteq \bigcup_{y\in\mathbb{Z}[\beta]\cap[0,1)} \partial\mathcal{S}(y)$ by Theorem 3.3 (iii), we have

$$\frac{\mu_\infty^c(\partial\mathcal{S}(x))}{\mu_\infty^c(X)} = \frac{\mu_\infty^c(\beta^{-n}\,\partial\mathcal{S}(x))}{\mu_\infty^c(\beta^{-n}X)} \le \frac{\mu_\infty^c\big(\bigcup_{y\in\mathbb{Z}[\beta]\cap[0,1)} \partial\mathcal{S}(y) \cap \beta^{-n}X\big)}{\mu_\infty^c(\beta^{-n}X)} \tag{3.22}$$

for all $n \in \mathbb{N}$. To get an upper bound for $\mu_\infty^c(\bigcup_{y\in\mathbb{Z}[\beta]\cap[0,1)} \partial\mathcal{S}(y) \cap \beta^{-n}X)$, let

$$R_k(v) = \{z \in T_\beta^{-k}(v) : \beta^k\,\mathcal{R}(z) \cap \partial\mathcal{R}(v) \neq \emptyset\} \qquad (v \in V). \tag{3.23}$$

Then, for each $y \in \mathbb{Z}[\beta] \cap [v, \widehat{v})$,

$$\begin{aligned}\partial\mathcal{S}(y) \times \delta_{\mathrm{f}}(\{0\}) \subseteq \partial\mathcal{R}(y) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{0\}) &= \big(\partial\mathcal{R}(v) + \delta_c(y-v)\big) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{0\}) \\ &\subseteq \delta_c(y-v) + \bigcup_{z\in R_k(v)} \beta^k\,\mathcal{R}(z) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{v-y\}).\end{aligned}$$

If $\beta^k\,\mathcal{R}(z) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{v-y\})$ is non-empty for a given $z \in R_k(v) \subseteq \beta^{-k}\mathbb{Z}[\beta]$, $v \in V$, then $y \in v - \beta^k(z + \mathbb{Z}[\beta])$ by Lemma 1.29, thus

$$\bigcup_{y\in\mathbb{Z}[\beta]\cap[0,1)} \partial\mathcal{S}(y) \subseteq \bigcup_{v\in V} \bigcup_{z\in R_k(v)} \bigcup_{y\in(v-\beta^k z+\beta^k\mathbb{Z}[\beta])\cap[v,\widehat{v})} \big(\delta_\infty^c(y-v) + \pi_\infty^c(\beta^k\,\mathcal{R}(z))\big).$$

Setting

$$C_{k,n,v}(z) = \#\big\{y \in (v-\beta^k z+\beta^k\mathbb{Z}[\beta])\cap[v,\widehat{v}) :\, \delta_\infty^c(y-v) \in \beta^{-n}X - \pi_\infty^c(\beta^k\,\mathcal{R}(z))\big\},$$

we have that

$$\mu_\infty^c\Bigg(\bigcup_{y\in\mathbb{Z}[\beta]\cap[0,1)} \partial\mathcal{S}(y) \cap \beta^{-n}X\Bigg) \le \sum_{v\in V} \sum_{z\in R_k(v)} \sum_{y\in C_{k,n,v}(z)} \mu_\infty^c\big(\pi_\infty^c(\beta^k\,\mathcal{R}(z))\big). \tag{3.24}$$

Now, we estimate the number of terms in the sums in (3.24). First, we have

$$\#R_k(v) = O(\alpha^k) \tag{3.25}$$

for some $\alpha < \beta$. Indeed, we can define, similarly to the boundary graph, a directed multiple graph with set of nodes V and $\#(T_\beta^{-1}(v) \cap [w, \widehat{w}))$ edges from v to w, $v, w \in V$. Then this graph is strongly connected and the number of paths of length k starting from $v \in V$ is $\#T_\beta^{-k}(v)$, whose order of growth is β^k, thus the spectral radius of the graph is β. Since the interior of $\mathcal{R}(0)$ is non-empty, we have $\beta^m\,\mathcal{R}(z) \cap \partial\mathcal{R}(0) = \emptyset$ for some $z \in T_\beta^{-m}(0)$, $m \in \mathbb{N}$. Let p be the corresponding path of length m from 0 to w, with $z \in [w, \widehat{w})$. Then there is $\alpha < \beta$ such that the number of paths of length k starting from $v \in V$ that avoid p is $O(\alpha^k)$, hence (3.25) holds.

From Lemma 1.32, we obtain that

$$\frac{C_{k,n,v}(z)}{\#\big\{x \in \mathbb{Z}[\beta] \cap [0,1) :\, \delta_\infty^c(x) \in \beta^{-n}\,X\big\}} \le \frac{3}{|N(\beta)|^k} \tag{3.26}$$

for all $z \in \beta^{-k}\mathbb{Z}[\beta]$, $v \in V$, for sufficiently large n. (The subtraction of $\pi_\infty^c(\beta^k\,\mathcal{R}(z))$ in the definition of $C_{k,n,v}(z)$ is negligible when n is large compared to k.) As $\delta_\infty^c(\mathbb{Z}[\beta] \cap [0,1))$ is a Delone set, we have

$$\#\big\{x \in \mathbb{Z}[\beta] \cap [0,1) :\, \delta_\infty^c(x) \in \beta^{-n}\,X\big\} = O\big(\mu_\infty^c(\beta^{-n}\,X)\big). \tag{3.27}$$

Finally, we use that

$$\mu_\infty^c\Big(\pi_\infty^c\big(\beta^k\,\mathcal{R}(z)\big)\Big) = O\bigg(\frac{|N(\beta)|^k}{\beta^k}\bigg) \tag{3.28}$$

for all $z \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$. Inserting (3.25)–(3.28) in (3.24) gives that

$$\frac{\mu_\infty^c\big(\bigcup_{y\in\mathbb{Z}[\beta]\cap[0,1)} \partial\mathcal{S}(y) \cap \beta^{-n}X\big)}{\mu_\infty^c(\beta^{-n}X)} \le c\,\frac{\alpha^k}{\beta^k}$$

for all $k \in \mathbb{N}$ and sufficiently large $n \in \mathbb{N}$, with some constant $c > 0$. Together with (3.22), this implies that $\mu_\infty^c(\partial\mathcal{S}(x)) \le c\,\frac{\alpha^k}{\beta^k}\,\mu_\infty^c(X)$ for all $k \in \mathbb{N}$, i.e., $\mu_\infty^c(\partial\mathcal{S}(x)) = 0$. This concludes the proof of Theorem 3.3 (ii).

3.5.4. Quadratic Pisot numbers. To prove Theorem 3.3 (v), let β be a quadratic Pisot number, and denote by z' the Galois conjugate of $z \in \mathbb{Q}(\beta)$. We show first that

$$\operatorname{sgn}(x_1' - y_1') = \operatorname{sgn}(\beta')\,\operatorname{sgn}(x' - y') \tag{3.29}$$

for all $x_1 \in T_\beta^{-1}(x) \cap \mathbb{Z}[\beta]$, $y_1 \in T_\beta^{-1}(y) \cap \mathbb{Z}[\beta]$, with $x, y \in \mathbb{Z}[\beta] \cap [0,1)$. Writing $x_1 = \frac{a+x}{\beta}$, $y_1 = \frac{b+y}{\beta}$ with $a, b \in \mathcal{A}$, we have $\operatorname{sgn}(x_1' - y_1') = \operatorname{sgn}(\beta')\,\operatorname{sgn}(a - b + x' - y')$. Write now $x = m\beta - \lfloor m\beta\rfloor$, $y = n\beta - \lfloor n\beta\rfloor$ with $m, n \in \mathbb{Z}$, and assume that $n = m + 1$. Then

$$a - b + x' - y' = a - b + \lfloor (m+1)\beta\rfloor - \lfloor m\beta\rfloor - \beta' \ge a - b + \lfloor\beta\rfloor - \beta'.$$

Recalling that $|\beta'| < 1$ since β is a Pisot number, we obtain that $a - b + x' - y' > 0$ if $b < \lfloor\beta\rfloor$. If $b = \lfloor\beta\rfloor$, then $y_1 < 1$ implies that $\lfloor\beta\rfloor + (m+1)\beta - \lfloor(m+1)\beta\rfloor < \beta$, thus $\lfloor m\beta\rfloor = \lfloor(m+1)\beta\rfloor - \lfloor\beta\rfloor - 1$, hence we also get that $\operatorname{sgn}(a - b + x' - y') = 1$. Since $\operatorname{sgn}(x' - y') = \operatorname{sgn}(n - m)$, we obtain that (3.29) holds in the case $n = m+1$, and we infer that (3.29) holds in the general case.

Inductively, we get that $\operatorname{sgn}(x_k' - y_k') = \operatorname{sgn}(\beta')^k\,\operatorname{sgn}(x' - y')$ for all $x_k \in T_\beta^{-k}(x) \cap \mathbb{Z}[\beta]$, $y_k \in T_\beta^{-k}(y) \cap \mathbb{Z}[\beta]$. Since $T_\beta(\mathbb{Z}[\beta] \cap [0,1)) \subseteq \mathbb{Z}[\beta] \cap [0,1)$, the sets $T_\beta^{-k}(x) \cap \mathbb{Z}[\beta]$, $x \in \mathbb{Z}[\beta] \cap [0,1)$, form a partition of $\mathbb{Z}[\beta] \cap [0,1)$ for each $k \in \mathbb{N}$. Renormalizing by $(\beta')^k$ and taking the Hausdorff limit shows that $\mathcal{S}(x)$ is an interval for each $x \in \mathbb{Z}[\beta] \cap [0,1)$ that meets $\bigcup_{y\in\mathbb{Z}[\beta]\cap[0,1)\setminus\{x\}} \mathcal{S}(y)$ only at its endpoints, i.e., $\mathcal{C}_{\mathrm{int}}$ is a tiling of $\mathbb{K}_\infty^c = \mathbb{R}$.

3.6. Equivalence between different tiling properties

3.6.1. Multiple tilings. We first recall the proof that $\mathcal{C}_{\mathrm{aper}}$ is a multiple tiling; see e.g. [IR06, KS12] for the unit case. Since $\delta_c(\mathbb{Z}[\beta^{-1}] \cap [0,1))$ is a Delone set and $\operatorname{diam}\mathcal{R}(x)$ is uniformly bounded, $\mathcal{C}_{\mathrm{aper}}$ is uniformly locally finite. Suppose that $\mathcal{C}_{\mathrm{aper}}$ is not a multiple tiling. Then there are integers $m_1 > m_2$ such that some set U of positive measure is covered at least m_1 times by elements of $\mathcal{C}_{\mathrm{aper}}$ and some point $\mathbf{z}$ lies in exactly m_2 tiles. Since the set $\bigcup_{x\in\mathbb{Z}[\beta^{-1}]\cap[0,1)} \partial\mathcal{R}(x)$ has measure zero, we can assume that U is contained in the complement of this set, and we can choose U to be open. For small $\varepsilon > 0$, the configuration of the tiles in a large neighbourhood of $\delta_c(x)$ does not depend on $x \in \mathbb{Z}[\beta^{-1}] \cap [0,\varepsilon)$. (In the parlance of [IR06, KS12], the collection $\mathcal{C}_{\mathrm{aper}}$ is quasi-periodic.) Therefore, each element of $\mathbf{z} + \delta_c(\mathbb{Z}[\beta^{-1}] \cap [0,\varepsilon))$ lies in exactly m_2 tiles. By the set equations in Theorem 3.1 (iii), $\beta^{-k}U$ is covered at least m_1 times for all $k \in \mathbb{N}$. Since

$\delta_c(\mathbb{Z}[\beta^{-1}] \cap [0,\varepsilon))$ is a Delone set and $\beta^{-k}U$ is arbitrarily large, we have a contradiction.

As $\tilde{\mathcal{C}}_{\text{aper}}$ is the restriction of $\mathcal{C}_{\text{aper}}$ to Z^c, it is a multiple tiling with same covering degree.

The multiple tiling property of $\mathcal{C}_{\text{ext}}$ also follows from that of $\mathcal{C}_{\text{aper}}$. Indeed, for $x, y \in \mathbb{Z}[\beta^{-1}]$, we have $(\delta(x) + \mathscr{X}) \cap \{y\} \times \mathbb{K}_\beta^c \neq \emptyset$ if and only if $y - x \in [0,1)$. Since

$$\big(\delta(x) + \mathscr{X}\big) \cap \{y\} \times \mathbb{K}_\beta^c = \{y\} \times \big(\delta_c(y) - \mathcal{R}(y-x)\big) = \delta(y) - \{0\} \times \mathcal{R}(y-x)$$

if $y - x \in [0,1)$ and $\mathbb{Z}[\beta^{-1}]$ is dense in $\mathbb{R}$, we obtain that $\mathcal{C}_{\text{ext}}$ is a multiple tiling with same covering degree as $\mathcal{C}_{\text{aper}}$. Again, $\tilde{\mathcal{C}}_{\text{ext}}$ being the restriction of $\mathcal{C}_{\text{ext}}$ to Z, it is a multiple tiling with same covering degree.

Almost the same proof as for $\mathcal{C}_{\text{aper}}$ shows that $\mathcal{C}_{\text{int}}$ is a multiple tiling. The collection $\mathcal{C}_{\text{int}}$ need not be quasi-periodic, but we have "almost quasi-periodicity" by Theorem 3.3 (iv). If $\mathbf{z} \in \mathbb{K}_\infty^c$ lies in exactly m_2 tiles, then for large $k \in \mathbb{N}$ each element of $\mathbf{z} + \delta_\infty^c(\beta^k\mathbb{Z}[\beta] \cap [0,\varepsilon))$ lies in at most m_2 tiles. As $\delta_\infty^c(\beta^k\mathbb{Z}[\beta] \cap [0,\varepsilon))$ is Delone, we get that $\mathcal{C}_{\text{int}}$ is a multiple tiling.

The relation between $\tilde{\mathcal{C}}_{\text{aper}}$ and $\mathcal{C}_{\text{int}}$ is similar to that between $\mathcal{C}_{\text{ext}}$ and $\mathcal{C}_{\text{aper}}$, as one is obtained from the other by intersection with a suitable "hyperplane". However, the proof that $\mathcal{C}_{\text{int}}$ has the same covering degree as $\tilde{\mathcal{C}}_{\text{aper}}$ needs a bit more attention than that for $\mathcal{C}_{\text{ext}}$. Let m be the covering degree of $\tilde{\mathcal{C}}_{\text{aper}}$ and choose $y_1, \ldots, y_m \in \mathbb{Z}[\beta] \cap [0,1)$ such that the interior U of $\bigcap_{i=1}^m \mathcal{R}(y_i)$ is non-empty; then $U \cap \mathcal{R}(x) = \emptyset$ for all $x \notin \{y_1, \ldots, y_m\}$. Set

$$\varepsilon = \min\big\{\widehat{x} - x : \, x \in \mathbb{Z}[\beta] \cap (-1,1),\ \delta_\infty^c(x) \in \pi_\infty^c\big(U - \mathcal{R}(0)\big)\big\},$$

with $\widehat{x} = 0$ for $x < 0$. As $\delta_{\mathrm{f}}(\mathbb{Z}[\beta] \cap [0,\varepsilon))$ is dense in $\overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$, there exists $z \in \mathbb{Z}[\beta] \cap [0,\varepsilon)$ such that $U \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{-z\}) \neq \emptyset$. Then the set

$$\tilde{U} = \delta_\infty^c(z) + \pi_\infty^c\big(U \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{-z\})\big)$$

is open. If $\mathcal{S}(x+z) \cap \tilde{U} \neq \emptyset$ for $x \in \mathbb{Z}[\beta] \cap [-z, 1-z)$, then we get from $\mathcal{S}(x+z) \subseteq \delta_\infty^c(x+z) + \pi_\infty^c(\mathcal{R}(0))$ that $\delta_\infty^c(x+z) \in \tilde{U} - \pi_\infty^c(\mathcal{R}(0))$, i.e., $\delta_\infty^c(x) \in \pi_\infty^c(U - \mathcal{R}(0))$. Therefore, we have $\widehat{x} - x \geq \varepsilon$, which ensures that $x \in [0, 1-z)$ because $x \in [-z, 0)$ would mean that $\widehat{x} - x = -x \leq z < \varepsilon$. Now, $x + z < x + \varepsilon \leq \widehat{x}$ implies that

$$\begin{aligned}\mathcal{R}(x) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{-z\}) &= \big(\mathcal{R}(x+z) - \delta_c(z)\big) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{-z\}) \\ &= \mathcal{S}(x+z) \times \delta_{\mathrm{f}}(\{0\}) - \delta_c(z).\end{aligned}$$

From $U \subseteq \mathcal{R}(y_i)$, we obtain that $\pi_\infty^c(U \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{-z\})) \subseteq \mathcal{S}(y_i + z) - \delta_\infty^c(z)$, thus

$$\tilde{U} \subseteq \mathcal{S}(y_i + z) \quad \text{for} \quad 1 \leq i \leq m.$$

We have already seen that $\mathcal{S}(x+z) \cap \tilde{U} = \emptyset$ for $x \in \mathbb{Z}[\beta] \cap [-z, 0)$. For $x \in \mathbb{Z}[\beta] \cap [0, 1-z) \setminus \{y_1, \ldots, y_m\}$, the disjointness of $\mathcal{S}(x+z)$ and $\tilde{U}$ follows from $U \cap \mathcal{R}(x) = \emptyset$. Thus, $\tilde{U}$ is a set of positive measure that is covered exactly m times, i.e., the covering degree of $\mathcal{C}_{\text{int}}$ is m.

We have proved that the collections $\mathcal{C}_{\text{ext}}$, $\tilde{\mathcal{C}}_{\text{ext}}$, $\mathcal{C}_{\text{aper}}$, $\tilde{\mathcal{C}}_{\text{aper}}$, and $\mathcal{C}_{\text{int}}$ are all multiple tilings with the same covering degree. The equivalence of Theorem 3.4 (i)–(iii) follows immediately.

3.6.2. Property (W). Several slightly different but equivalent definitions of weak finiteness can be found in [Hol96, Aki02, Sid03, ARS04]. Our definition of (W), which is called (H) in [ARS04], is essentially due to Hollander. By [ARS04], this property holds for each quadratic Pisot number, for each cubic Pisot unit, as well as for each $\beta > 1$ satisfying $\beta^d = t_1\beta^{d-1}+t_2\beta^{d-2}+\cdots+t_{d-1}\beta+t_d$ for some $t_1, \ldots, t_d \in \mathbb{Z}$ with $t_1 > \sum_{k=2}^d |t_k|$. Other classes of numbers giving tilings and thus satisfying (W) were found by [BK05, BBK06].

An immediate consequence of (3.5) is that, for $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$,

$$\delta_c(0) \in \mathcal{R}(x) \quad \text{if and only if} \quad x \in P = \mathrm{Pur}(\beta) \cap \mathbb{Z}[\beta].$$

(Note that we cannot have $T_\beta^n(x) = x$ for $x \in \mathbb{Z}[\beta^{-1}] \setminus \mathbb{Z}[\beta]$.) For general $z \in \mathbb{Z}[\beta^{-1}] \cap [0,\infty)$, let $n \in \mathbb{N}$ be such that $y + \beta^{-n}z < \widehat{y}$ for all $y \in P$. Then we have

$$\delta_c(z) \in \mathcal{R}(x) \quad \text{if and only if} \quad x \in T_\beta^n(P + \beta^{-n}z). \tag{3.30}$$

To prove (3.30), let first $x = T_\beta^n(y + \beta^{-n}z)$, $y \in P$, and let $k \geq 1$ be such that $T_\beta^k(y) = y$. Then $y+\beta^{-n}z < \widehat{y}$ implies that $T_\beta^{n+jk}(y+\beta^{-n-jk}z) = T_\beta^n(y+\beta^{-n}z) = x$ for all $j \in \mathbb{N}$; cf. the proof of Theorem 3.1 (v) in Section 3.4. Thus we have $\delta_c(\beta^{n+jk}(y+\beta^{-n-jk}z)) \in \mathcal{R}(x)$, i.e., $\delta_c(z) \in \mathcal{R}(x) - \delta_c(\beta^{n+jk}y)$. Taking the limit for $j \to \infty$, we conclude that $\delta_c(z) \in \mathcal{R}(x)$.

Let now $\delta_c(z) \in \mathcal{R}(x)$, $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$, $z \in \mathbb{Z}[\beta^{-1}] \cap [0,\infty)$. For each $k \in \mathbb{N}$, there is an $x_k \in T_\beta^{-k}(x)$ such that $\delta_c(\beta^{-k}z) \in \mathcal{R}(x_k)$. Then the set $\{\delta_c(\beta^{-k}z - x_k) : k \in \mathbb{N}\}$ is bounded and thus finite by Lemma 1.23. Choose $y \in \mathbb{Z}[\beta^{-1}]$ such that $x_k - \beta^{-k}z = y$ for infinitely many $k \in \mathbb{N}$. Let j and k be two successive elements of the set $\{k \in \mathbb{N} : x_k - \beta^{-k}z = y\}$, with j large enough such that $y + \beta^{-j}z < \widehat{y}$. Then $T_\beta^{k-j}(y + \beta^{-k}z) = y + \beta^{-j}z$ and $T_\beta^{k-j}(y + \beta^{-k}z) = T_\beta^{k-j}(y) + \beta^{-j}z$ thus $T_\beta^{k-j}(y) = y$, which implies that $y \in P$ and $T_\beta^j(y + \beta^{-j}z) = T_\beta^j(x_j) = x$. We infer that $x \in T_\beta^n(P + \beta^{-n}z)$ for all $n \in \mathbb{N}$ such that $y + \beta^{-n}z < \widehat{y}$ for all $y \in P$, thus (3.30) holds.

We use (3.30) to prove the equivalence between (W) and the tiling property of $\mathcal{C}_{\mathrm{aper}}$, similarly to [Aki02, KS12]. If $\mathcal{C}_{\mathrm{aper}}$ is a tiling, then $\mathcal{R}(0)$ contains an exclusive point. Since $\delta_c(\mathbb{Z}[\beta^{-1}] \cap [0,\infty))$ is dense in $\mathbb{K}_\beta^c$, we have thus an exclusive point $\delta_c(z) \in \mathcal{R}(0)$ with $z \in \mathbb{Z}[\beta^{-1}] \cap [0,\infty)$. By (3.30), this implies that $T_\beta^n(P + \beta^{-n}z) = \{0\}$ for all $n \in \mathbb{N}$ satisfying $x + \beta^{-n}z < \widehat{x}$ for all $x \in P$, in particular $T_\beta^n(\beta^{-n}z) = 0$ because $0 \in P$. Hence, (W) holds with $y = \beta^{-n}z$ (independently from $x \in P$).

Now, we assume that (W) holds and construct an exclusive point $\delta_c(z) \in \mathcal{R}(0)$ of $\mathcal{C}_{\mathrm{aper}}$, with $z \in \mathbb{Z}[\beta^{-1}] \cap [0,\infty)$, as in [Aki02, KS12]. Note first that (W) implies that

$$\forall\, x \in P,\ \varepsilon > 0\ \exists\, y \in [0,\varepsilon),\ n \in \mathbb{N} :\ T_\beta^n(x+y) = T_\beta^n(y) = 0. \tag{3.31}$$

Indeed, if $x = \max\{T_\beta^j(x) : j \in \mathbb{N}\}$, $T_\beta^k(x) = x$, and $T_\beta^n(x+y) = T_\beta^n(y) = 0$, then we have $T_\beta^{\ell k-j}(T_\beta^j(x) + \beta^{j-\ell k}y) = x + y$ and thus $T_\beta^{n+\ell k-j}(T_\beta^j(x) + \beta^{j-\ell k}y) = 0 = T_\beta^{n+\ell k-j}(\beta^{j-\ell k}y)$ for all $\ell \geq 1$, $0 \leq j < k$, which proves (3.31); see also [ARS04, Lemma 1].

If $P = \{0\}$, then $\delta_c(0)$ is an exclusive point. Otherwise, let $P = \{0, x_1, \ldots, x_h\}$ and set

$$z = \beta^{k_1+k_2+\cdots+k_h} y_1 + \beta^{k_2+\cdots+k_h} y_2 + \cdots + \beta^{k_h} y_h,$$

where y_j and k_j are defined recursively for $1 \le j \le h$ with the following properties:

- $T_\beta^{k_j}\big(T_\beta^{k_1+\cdots+k_{j-1}}(x_j + \sum_{i=1}^{j-1} y_i\, \beta^{-k_1-\cdots-k_{i-1}}) + y_j\big) = T_\beta^{k_j}(y_j) = 0$,
- $T_\beta^{k_1+\cdots+k_j}\big(x_{j+1} + \sum_{i=1}^{j} y_i\, \beta^{-k_1-\cdots-k_{i-1}}\big) \in P$, if $j < h$,
- $x + \sum_{i=\ell}^{j} y_i\, \beta^{-k_\ell-\cdots-k_{i-1}} < \widehat{x}$ for all $x \in P$, $1 \le \ell \le j$.

Then we have

$$\begin{aligned}
&T_\beta^{k_1+\cdots+k_h}(x_j + \beta^{-k_1-\cdots-k_h} z) = T_\beta^{k_1+\cdots+k_h}\big(x_j + \textstyle\sum_{i=1}^{h} y_i\, \beta^{-k_1-\cdots-k_{i-1}}\big)\\
&= T_\beta^{k_j+\cdots+k_h}\Big(T_\beta^{k_1+\cdots+k_{j-1}}\big(x_j + \textstyle\sum_{i=1}^{j-1} y_i\, \beta^{-k_1-\cdots-k_{i-1}}\big)\\
&\qquad + y_j + \textstyle\sum_{i=j+1}^{h} y_i\, \beta^{-k_j-\cdots-k_{i-1}}\Big)\\
&= T_\beta^{k_{j+1}+\cdots+k_h}\big(\textstyle\sum_{i=j+1}^{h} y_i\, \beta^{-k_{j+1}-\cdots-k_{i-1}}\big)\\
&= T_\beta^{k_{j+2}+\cdots+k_h}\big(\textstyle\sum_{i=j+2}^{h} y_i\, \beta^{-k_{j+2}-\cdots-k_{i-1}}\big)\\
&= \cdots = 0
\end{aligned}$$

for $0 \le j \le h$, with $x_0 = 0$. By (3.30), $\delta_c(z)$ is an exclusive point of $\mathcal{C}_{\mathrm{aper}}$ (in $\mathcal{R}(0)$).

Therefore, we have proved that Theorem 3.4 (iv) is equivalent to (i)–(iii).

3.6.3. Boundary graph. We study properties of the boundary graph defined in Section 3.1 and prove the tiling condition Theorem 3.4 (v). First note that the boundary graph is finite since, for each node $[v, x, w]$, $\delta_\infty^c(x)$ is contained in the intersection of the Delone set $\delta_\infty^c(\mathbb{Z}[\beta] \cap (-1, 1))$ with the bounded set $\pi_\infty^c(\mathcal{R}(0) - \mathcal{R}(0))$.

Next, we show that the labels of the infinite paths in the boundary graph provide pairs of expansions exactly for the points that lie in $\mathcal{R}(x) \cap \mathcal{R}(y)$ for some $x, y \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$, $x \ne y$. When $\mathcal{C}_{\mathrm{aper}}$ is a tiling, these points are exactly the boundary points of $\mathcal{R}(x)$, $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$, hence the name "boundary graph". (In general, "intersection graph" might be a better name.) If $x \in \mathbb{Z}[\beta^{-1}] \cap [v, \widehat{v})$, $y \in \mathbb{Z}[\beta^{-1}] \cap [w, \widehat{w})$, $v, w \in V$, $x \ne y$, then

$$\mathcal{R}(x) \cap \mathcal{R}(y) \ne \emptyset \quad \Leftrightarrow \quad [v, y-x, w] \text{ is a node of the boundary graph,} \tag{3.32}$$

and

$$\mathbf{z} \in \mathcal{R}(x) \cap \mathcal{R}(y) \quad \Leftrightarrow \quad \mathbf{z} = \delta_c(x) + \sum_{k=0}^{\infty} \delta_c(a_k \beta^k) = \delta_c(y) + \sum_{k=0}^{\infty} \delta_c(b_k \beta^k) \tag{3.33}$$

where $(a_0, b_0)(a_1, b_1) \cdots$ is the sequence of labels of an infinite path starting in $[v, y-x, w]$.

For $x \in \mathbb{Z}[\beta^{-1}] \cap [v, \widehat{v})$, $y \in \mathbb{Z}[\beta^{-1}] \cap [w, \widehat{w})$, we have $\mathcal{R}(x) \cap \mathcal{R}(y) \ne \emptyset$ if and only if

$$\delta_c(0) \in \mathcal{R}(x) - \mathcal{R}(y) = \mathcal{R}(v) - \mathcal{R}(w) + \delta_c(w - v) - \delta_c(y - x),$$

i.e., $\delta_c(y - x) \in \mathcal{R}(v) - \mathcal{R}(w) + \delta_c(w - v) \subseteq \overline{\delta_c(\mathbb{Z}[\beta])}$ and thus $y - x \in \mathbb{Z}[\beta]$ by Lemma 1.29. If moreover $x \ne y$, this is equivalent to $[v, y-x, w]$ being a node of the boundary graph, which proves (3.32). Let $\mathbf{z} \in \mathcal{R}(x) \cap \mathcal{R}(y)$. Using Theorem 3.1 (iii), we have

$$\mathcal{R}(x)\cap\mathcal{R}(y) = \bigcup_{\substack{x_1\in T_\beta^{-1}(x)\\ y_1\in T_\beta^{-1}(y)}} \beta\,\mathcal{R}(x_1)\cap\beta\,\mathcal{R}(y_1) = \bigcup_{[v,y-x,w]\overset{(a_0,b_0)}{\to}[v_1,\frac{b_0-a_0+y-x}{\beta},w_1]} \beta\left(\mathcal{R}\big(\tfrac{a_0+x}{\beta}\big)\cap\mathcal{R}\big(\tfrac{b_0+y}{\beta}\big)\right),$$

where the transitions are edges in the boundary graph, with $\frac{a_0+x}{\beta} \in [v_1, \widehat{v}_1)$, $\frac{b_0+y}{\beta} \in [w_1, \widehat{w}_1)$. Thus we have $\mathbf{z} \in \beta\,(\mathcal{R}(x_1) \cap \mathcal{R}(y_1))$ for some $x_1 = \frac{a_0+x}{\beta} \in T_\beta^{-1}(x)$, $y_1 = \frac{b_0+y}{\beta} \in T_\beta^{-1}(y)$, and $[v, y-x, w] \overset{(a_0,b_0)}{\to} [v_1, y_1-x_1, w_1]$ is an edge in the boundary graph. Iterating this observation, we get a path in the boundary graph labelled by $(a_0, b_0)(a_1, b_1)\cdots$ such that, for each $k \in \mathbb{N}$, $\mathbf{z} \in \beta^k\,(\mathcal{R}(x_k) \cap \mathcal{R}(y_k))$ with $x_k = \big(\sum_{j=0}^{k-1} a_j\beta^j + x\big)\,\beta^{-k} \in T_\beta^{-k}(x)$, $y_k = \big(\sum_{j=0}^{k-1} b_j\beta^j + y\big)\,\beta^{-k} \in T_\beta^{-k}(y)$. Since $\lim_{k\to\infty} \beta^k\,\mathcal{R}(x_k) = \delta_c(x) + \sum_{j=0}^{\infty} \delta_c(a_j\beta^j)$ and $\lim_{k\to\infty} \beta^k\,\mathcal{R}(y_k) = \delta_c(y) + \sum_{j=0}^{\infty} \delta_c(b_j\beta^j)$, we obtain one direction of (3.33). The other direction of (3.33) is proved similarly.

Next, we prove that $\mathcal{C}_{\mathrm{aper}}$ is a tiling if and only if $\varrho < \beta$, where ϱ denotes the spectral radius of the boundary graph. We have seen that, for $x \in \mathbb{Z}[\beta^{-1}] \cap [v, \widehat{v})$, $y \in \mathbb{Z}[\beta^{-1}] \cap [w, \widehat{w})$ with $\mathcal{R}(x) \cap \mathcal{R}(y) \neq 0$, each path of length k starting from the node $[v, y-x, w]$ in the boundary graph corresponds to a pair $x_k \in T_\beta^{-k}(x)$, $y_k \in T_\beta^{-k}(y)$ with $\mathcal{R}(x_k) \cap \mathcal{R}(y_k) \neq \emptyset$. As, for each $\tilde{x} \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$, there is a bounded number of $\tilde{y} \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$ such that $\mathcal{R}(x) \cap \mathcal{R}(y) \neq \emptyset$, the number of these paths gives, up to a multiplicative constant, the number of subtiles $\beta^k\,\mathcal{R}(x_k)$ of $\mathcal{R}(x)$ that meet $\bigcup_{y\in\mathbb{Z}[\beta^{-1}]\cap[0,1)\setminus\{x\}} \mathcal{R}(y)$. Hence we have $\mu_c\big(\mathcal{R}(x) \cap \bigcup_{y\in\mathbb{Z}[\beta^{-1}]\cap[0,1)\setminus\{x\}} \mathcal{R}(y)\big) \leq P(k)\,\varrho^k\beta^{-k}$ for all $k \in \mathbb{N}$, with some polynomial $P(k)$. If $\varrho < \beta$, this yields that $\mathcal{C}_{\mathrm{aper}}$ is a tiling. On the other hand, if $\mathcal{C}_{\mathrm{aper}}$ is a tiling, then $\partial\mathcal{R}(x) = \mathcal{R}(x) \cap \bigcup_{y\in\mathbb{Z}[\beta^{-1}]\cap[0,1)\setminus\{x\}} \mathcal{R}(y)$, thus the number of paths of length k from $[v, y-x, w]$ is bounded by a constant times $R_k(v)$, with $R_k(v)$ as in (3.23). Since $\#R_k(v) = O(\alpha^k)$ with $\alpha < \beta$ by (3.25), we have that $\varrho \leq \alpha < \beta$. Therefore, Theorem 3.4 (v) is equivalent to (i)–(iii).

The equivalence between Theorem 3.4 (i)–(ii) and (v) can also be extended to multiple tilings. To this end, one defines a generalisation of the boundary graph that recognises all points that lie in m tiles (instead of 2 tiles). Then the spectral radius of this graph is less than β if and only if the covering degree of $\mathcal{C}_{\mathrm{aper}}$ is less than m.

3.6.4. Periodic tiling with Rauzy fractals. The last of our equivalent tiling conditions is that of the periodic tiling, under the condition that (QM) holds. This condition is satisfied when the size of V is equal to the degree of the algebraic number β; the following examples show that (QM) can be true or false when $\#V > \deg(\beta)$.

Example 3.7. Let $\beta > 1$ satisfy $\beta^3 = t\beta^2 - \beta + 1$ for some integer $t \geq 2$. Then

$$T_\beta(1^-) = \beta - (t-1) = \tfrac{(t-1)\beta^2+1}{\beta^3}, \quad T_\beta^2(1^-) = \tfrac{(t-1)\beta^2+1}{\beta^2} - (t-1) = \tfrac{1}{\beta^2},$$
$$T_\beta^3(1^-) = \tfrac{1}{\beta}, \quad T_\beta^4(1^-) = 1,$$

thus $\widehat{V} = \{1, \beta-(t-1), \beta^2-(t-1)\beta-(t-1), \beta^2-t\beta+1\}$, and $L = \langle \beta-t, \beta^2-t\beta\rangle_{\mathbb{Z}}$. Therefore, (QM) holds.

EXAMPLE 3.8. Let $\beta > 1$ satisfy $\beta^3 = t\beta^2 + (t+1)\beta + 1$ for some integer $t \geq 0$. (For $t = 0$, β is the smallest Pisot number.) Then

$$T_\beta(1^-) = \beta - (t+1) = \tfrac{t\beta+1}{\beta^4}, \quad T_\beta^2(1^-) = \tfrac{t\beta+1}{\beta^3}, \quad T_\beta^3(1^-) = \tfrac{t\beta+1}{\beta^2},$$
$$T_\beta^4(1^-) = \tfrac{t\beta+1}{\beta} - t = \tfrac{1}{\beta}, \quad T_\beta^5(1^-) = 1,$$

thus $\widehat{V} = \{1, \beta-(t+1), \beta^2-(t+1)\beta, -\beta^2+(t+1)\beta+1, \beta^2-t\beta-(t+1)\}$. Since

$$1 = \big(1-\beta^2+(t+1)\beta\big) + \big(\beta^2-(t+1)\beta\big) = \big(1-T_\beta^2(1^-)\big) + \big(1-T_\beta^3(1^-)\big) \in L,$$

we have $\widehat{V} \subseteq L$ and thus $L = \mathbb{Z}[\beta]$, hence (QM) does not hold. According to [EI05] (see also [EIR06]), the central tile $\mathcal{R}(0)$ associated with the smallest Pisot number β cannot tile periodically its representation space $\mathbb{K}_\beta^c = \mathbb{C}$.

The central tile $\mathcal{R}(0)$ is closely related to the set of non-negative *β-integers*

$$\mathbb{N}_\beta = \bigcup_{k\geq 0} \beta^k\, T_\beta^{-k}(0),$$

as $\mathcal{R}(0) = \overline{\delta_c(\mathbb{N}_\beta)}$. We know from [Thu89, Fab95, FGK03] that the sequence of distances between consecutive elements of $\mathbb{N}_\beta$ is the fixed point of the *β-substitution* σ, which can be defined on the alphabet $\widehat{V}$ by

$$\sigma(x) = \underbrace{1\,1\cdots 1}_{\lceil T_\beta(x^-)\rceil - 1\text{ times}}\ T_\beta(x^-) \qquad (x \in \widehat{V}).$$

More precisely, we have $\mathbb{N}_\beta = \big\{\sum_{k=0}^{m-1} w_k : m \in \mathbb{N}\big\}$, where $w_0 w_1 \cdots \in \widehat{V}^{\mathbb{N}}$ is the infinite word starting with $\sigma^k(1)$ for all $k \in \mathbb{N}$. Similarly to [IR06, Proposition 3.4], we obtain that

$$L + \mathbb{N}_\beta = \Big\{\sum_{v\in V} c_v\,\widehat{v} : c_v \in \mathbb{Z},\ \sum_{v\in V} c_v \geq 0\Big\}, \tag{3.34}$$

using that $L = \{\sum_{v\in V} c_v\,\widehat{v} :\ c_v \in \mathbb{Z},\ \sum_{v\in V} c_v = 0\}$, which implies that

$$L + \sum_{k=0}^{m-1} w_k = \Big\{\sum_{v\in V} c_v\,\widehat{v} :\ c_v \in \mathbb{Z},\ \sum_{v\in V} c_v = m\Big\}$$

for all $n \in \mathbb{N}$. Next, we prove that

$$\delta_c(L) + \mathcal{R}(0) = Z^c. \tag{3.35}$$

If (QM) does not hold, then $\delta_c(L)$ is dense in $\overline{\delta_c(\mathbb{Z}[\beta])} = Z^c$, hence (3.35) follows from the fact that $\mathcal{R}(0)$ has non-empty interior. If (QM) holds, then it is sufficient to prove that

$$\overline{\delta_c(L+\mathbb{N}_\beta)} = \overline{\delta_c(\mathbb{Z}[\beta])}, \tag{3.36}$$

as $\delta_c(L)$ is a lattice in Z^c by Lemma 1.31 and $\mathcal{R}(0)$ is compact. Since $\widehat{V}$ spans $\mathbb{Z}[\beta]$, we can write each $x \in \mathbb{Z}[\beta]$ as $x = \sum_{v\in V} c_v\,\widehat{v}$, with $c_v \in \mathbb{Z}$. By (QM), we have $\beta^k \notin L$ for infinitely many $k \in \mathbb{N}$, thus $x + (\sum_{v\in V} c_v)\beta^k \in L + \mathbb{N}_\beta$ or $x - (\sum_{v\in V} c_v)\beta^k \in L+\mathbb{N}_\beta$ for these k. Since $\lim_{k\to\infty} \delta_c\big(x \pm (\sum_{v\in V} c_v)\beta^k\big) = \delta_c(x)$, we obtain that (3.36) and thus (3.35) holds.

Throughout the rest of the subsection, assume that (QM) holds. Then we have

$$\mathbb{N}_\beta \cap L = \{0\}$$

because $\sum_{k=0}^{m-1} w_k \in L$ for some $m \geq 1$ implies that

$$\Big\{ \sum_{v \in V} c_v\, \widehat{v} : c_v \in \mathbb{Z}, \sum_{v \in V} c_v = m \Big\} \subseteq L,$$

in particular $m\,\widehat{v} \in L$ for all $v \in V$, contradicting (QM). We immediately obtain that

$$\mathbb{N}_\beta \cap (x + \mathbb{N}_\beta) = \emptyset \quad \text{for all} \quad x \in L \setminus \{0\}, \tag{3.37}$$

i.e., $\{x + \mathbb{N}_\beta : x \in L\}$ forms a partition of $L + \mathbb{N}_\beta$. It is natural to expect from this partition property that $\mathcal{C}_{\mathrm{per}} = \{\delta_c(x) + \overline{\delta_c(\mathbb{N}_\beta)} : x \in L\}$ is a tiling, but this may not be true due to the effects of taking the closure. We can only prove that the tiling property of $\mathcal{C}_{\mathrm{per}}$ is equivalent to that of $\mathcal{C}_{\mathrm{aper}}$, similarly to [IR06, Proposition 3.5] and [Sin06b, Proposition 6.72 (v)].

Suppose that $\mathcal{C}_{\mathrm{aper}}$ is a tiling. From (3.35), we know that $\mathcal{C}_{\mathrm{per}}$ covers Z^c. Consider $\mathcal{R}(0) \cap (\delta_c(x) + \mathcal{R}(0))$ for some $x \in L \setminus \{0\}$, and assume w.l.o.g. that $x > 0$. As $\beta^{-k}\,\mathcal{R}(0) = \bigcup_{y \in T_\beta^{-k}(0)} \mathcal{R}(y)$, we have that

$$\beta^{-k} \Big(\mathcal{R}(0) \cap \big(\delta_c(x) + \mathcal{R}(0)\big) \Big) = \bigcup_{y,z \in T_\beta^{-k}(0)} \Big(\mathcal{R}(y) \cap \big(\delta_c(\beta^{-k} x) + \mathcal{R}(z)\big) \Big).$$

If $z + \beta^{-k} x < \widehat{z}$, then $\delta_c(\beta^{-k}x) + \mathcal{R}(z) = \mathcal{R}(z + \beta^{-k}x)$. Since $y \neq z + \beta^{-k}x$ for all $y, z \in T_\beta^{-k}(0)$ by (3.37), the tiling property of $\mathcal{C}_{\mathrm{aper}}$ implies that $\mu_c(\mathcal{R}(y) \cap \mathcal{R}(z + \beta^{-k}x)) = 0$ (if $z + \beta^{-k}x < 1$). Therefore, only tiles $\delta_c(\beta^{-k}x) + \mathcal{R}(z)$ with $z + \beta^{-k}x \geq \widehat{z}$ can contribute to the measure of $\mathcal{R}(0) \cap (\delta_c(x) + \mathcal{R}(0))$. The inequality $z + \beta^{-k}x \geq \widehat{z}$ is equivalent to $z \in [\widehat{v} - \beta^{-k}x, \widehat{v})$ for some $v \in V$. As $\beta^k\, T_\beta^{-k}(0) \subseteq \mathbb{N}_\beta$ and the distances between consecutive elements in $\mathbb{N}_\beta$ are in $\widehat{V}$, there are at most $(\#V)\, x \,/ \min \widehat{V}$ numbers $z \in T_\beta^{-k}(0)$ satisfying this inequality. This implies that

$$\mu_c\Big(\mathcal{R}(0) \cap \big(\delta_c(x) + \mathcal{R}(0)\big)\Big) \leq \frac{(\#V)\, x}{\min \widehat{V}}\, \mu_c\big(\beta^k\, \mathcal{R}(0)\big) = \frac{(\#V)\, x}{\min \widehat{V}}\, \mu_c\big(\mathcal{R}(0)\big)\, \beta^{-k}$$

for all $k \in \mathbb{N}$, hence $\mathcal{R}(0) \cap (\delta_c(x) + \mathcal{R}(0))$ has measure zero, thus $\mathcal{C}_{\mathrm{per}}$ is a tiling.

If $\mathcal{C}_{\mathrm{aper}}$ is not a tiling, then it covers $\mathbb{K}_\beta^c$ at least twice, and a similar proof as for the tiling property shows that $\mathcal{R}(0) \cap \bigcup_{x \in L \setminus \{0\}} (\delta_c(x) + \mathcal{R}(0))$ has positive measure. Hence, we have proved that Theorem 3.4 (vi) is equivalent to (i), which concludes the proof of Theorem 3.4.

With some more effort, the proof above can be adapted to show that $\mathcal{C}_{\mathrm{per}}$ is a multiple tiling with same covering degree as $\mathcal{C}_{\mathrm{aper}}$.

A patch of the periodic tiling induced by $\beta^3 = 2\beta^2 - \beta + 1$, which satisfies (QM) by Example 3.7, is depicted in Figure 3.6.

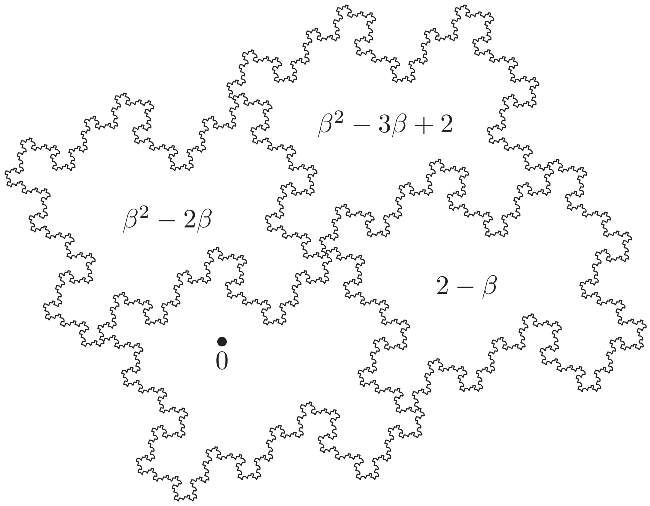


FIGURE 3.6. Patch of the periodic tiling $\mathcal{R}(0)+\delta_c(\langle\beta-2, \beta^2-2\beta\rangle_{\mathbb{Z}})$ induced by $\beta^3 = 2\beta^2 - \beta + 1$.

3.7. Gamma function

3.7.1. Proof of Theorem 3.5. To prove Theorem 3.5, suppose first that (3.12) does not hold. Then there exists $y \in \mathbb{Z}_{N(\beta)} \cap [0,1)$ with $\delta(y) \notin \mathscr{X}$ and

$$y < \inf\Big(\{1\} \cup \bigcup_{v\in V} \big\{x \in \mathbb{Q} \cap [v, \widehat{v}) : \delta_\infty^c(v-x) \in \pi_\infty^c\big(Z^c \setminus \mathcal{R}(v)\big)\big\}\Big). \tag{3.38}$$

Let $v \in V$ be such that $y \in [v, \widehat{v})$. Then $\delta_c(y) \notin \delta_c(v) - \mathcal{R}(v)$ and thus $\delta_c(v-y) \in Z^c \setminus \mathcal{R}(v)$ because $\delta_c(\mathbb{Z}_{N(\beta)} \cup V) \subseteq Z^c$. This contradicts (3.38), hence (3.12) holds.

Assume now that $\overline{\delta_{\mathrm{f}}(\mathbb{Q})} = \mathbb{K}_{\mathrm{f}}$. We want to prove the opposite inequality of (3.12). Since $\gamma(\beta) \le 1$, the inequality is clearly true if $\{x \in \mathbb{Q} \cap [v, \widehat{v}) : \delta_\infty^c(v-x) \in \pi_\infty^c(Z^c \setminus \mathcal{R}(v))\} = \emptyset$ for all $v \in V$. Otherwise, we show that arbitrarily close to each $x \in \mathbb{Q} \cap [v, \widehat{v})$ with $\delta_\infty^c(v-x) \in \pi_\infty^c(Z^c \setminus \mathcal{R}(v))$, we can find $y \in \mathbb{Z}_{N(\beta)}$ with $\delta(y) \notin \mathscr{X}$. Indeed, for sufficiently small $\varepsilon > 0$, we have

$$\delta_\infty^c(v-y) \in \pi_\infty^c(Z^c \setminus \mathcal{R}(v)) \quad \text{for all} \quad y \in \mathbb{Q} \cap (x, x+\varepsilon)$$

because $\pi_\infty^c(Z^c \setminus \mathcal{R}(v))$ is open. By [ABBS08, Lemma 4.7], we have

$$\overline{\delta\big(\mathbb{Z}_{N(\beta)} \cap (x, x+\varepsilon)\big)} = \overline{\delta_\infty\big(\mathbb{Q} \cap (x, x+\varepsilon)\big)} \times \overline{\delta_{\mathrm{f}}\big(\mathbb{Z}_{N(\beta)}\big)}.$$

The set

$$\{\mathbf{z} \in Z^c \setminus \mathcal{R}(v) : \pi_\infty^c(\mathbf{z}) \in \mathbb{Q} \cap (x, x+\varepsilon)\} \subseteq \delta_\infty^c\big(\mathbb{Q} \cap (x, x+\varepsilon)\big) \times \overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$$

is non-empty and open in $\delta_\infty^c(\mathbb{Q}) \times \mathbb{K}_{\mathrm{f}}$. Since $\overline{\delta_{\mathrm{f}}(\mathbb{Q})} = \mathbb{K}_{\mathrm{f}}$ implies that $\overline{\delta_{\mathrm{f}}(\mathbb{Z}_{N(\beta)})} = \overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$, we obtain that this set contains some $\delta_c(y)$ with $y \in \mathbb{Z}_{N(\beta)} \cap (x, x+\varepsilon)$. Then we have $\delta(y) \notin \mathscr{X}$. Since ε can be chosen arbitrary small, this concludes the proof of Theorem 3.5.

3.7.2. Boundary in the quadratic case. Let now β be a quadratic Pisot number. Then we show that the boundary of $\mathcal{R}(x)$ is simply the intersection with two of its neighbours. More precisely, for each $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$, we have that

$$\partial\mathcal{R}(x) = \mathcal{R}(x) \cap \Big(\mathcal{R}\big(x+\beta-\lfloor x+\beta\rfloor\big) \cup \mathcal{R}\big(x-\beta-\lfloor x-\beta\rfloor\big)\Big). \tag{3.39}$$

There may be other neighbours of $\mathcal{R}(x)$, but they meet $\mathcal{R}(x)$ only in points that also lie in $\mathcal{R}(x \pm \beta - \lfloor x \pm \beta\rfloor)$.

To prove (3.39), let $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$, and set $y = x+\beta-\lfloor x+\beta\rfloor$, $z = x-\beta-\lfloor x-\beta\rfloor$, $\varepsilon = \min\{\widehat{x}-x, \widehat{y}-y, \widehat{z}-z\}$. For each $u \in \mathbb{Z}[\beta] \cap [0,\varepsilon)$, we have

$$\mathcal{R}(x) \cap \mathbb{K}_\infty^c \times \delta_{\mathrm{f}}(\{-u\}) = \big(\mathcal{S}(x+u) - \delta_\infty^c(u)\big) \times \delta_{\mathrm{f}}(\{-u\}), \tag{3.40}$$

and $\mathcal{S}(x+u)$ is an interval by Theorem 3.3 (v). Since $\overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta] \cap [0,\varepsilon))} = \overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$ and $\mathcal{R}(x)$ is the closure of its interior, each $\mathbf{z} \in \partial\mathcal{R}(x)$ can be approximated by endpoints of intervals $(\mathcal{S}(x+u) - \delta_\infty^c(u)) \times \delta_{\mathrm{f}}(\{-u\})$, $u \in \mathbb{Z}[\beta] \cap [0,\varepsilon)$. By the proof of Theorem 3.3 (v) in Section 3.5.4 and since $y+u < \widehat{y}$, $z+u < \widehat{z}$, each endpoint of $\mathcal{S}(x+u)$ lies also in $\mathcal{S}(y+u)$ or $\mathcal{S}(z+u)$. As (3.40) still holds if we replace x by y or z, we obtain that $\mathbf{z} \in \mathcal{R}(y) \cup \mathcal{R}(z)$.

3.7.3. Pruned boundary graph. Formula (3.39) suggests the following definition: the *pruned boundary graph* of a quadratic Pisot number β is the subgraph of the boundary graph that is induced by the set of nodes $[v,x,w]$ with $x \in \pm\{\beta-\lfloor\beta\rfloor, \lceil\beta\rceil-\beta\}$.

By (3.39), we can replace the boundary graph by its pruned version in (3.32) and (3.33). This allows us to simplify some arguments of [ABBS08, Section 5].

In the following, let $\beta^2 = a\beta + b$ with $a \ge b \ge 1$. Then we have $V = \{0, v\}$ with $v = \beta - a$. We do not consider the case of negative b because we know from [Aki98, Proposition 5] that $\mathrm{Pur}(\beta) \cap \mathbb{Q} = \{0\}$ when β has a positive real conjugate, which implies that $\gamma(\beta) = 0$.

For the description of the pruned boundary graph, we have to distinguish two cases:

- If $2b \le a$, then the transitions of the pruned boundary graph are

$$\begin{aligned}
{[v, v{-}1, 0]} &\xrightarrow{(d,d+a-b+1)} [0, 1{-}v, v] && (0 \le d < b),\\
{[0, 1{-}v, v]} &\xrightarrow{(d,d-a+b-1)} [v, v{-}1, 0] && (a-b < d \le a),\\
{[0, v, v], [v, v, v]} &\xrightarrow{(d,d+a-b)} [0, 1{-}v, v] && (0 \le d < b),\\
{[v, -v, 0], [v, -v, v]} &\xrightarrow{(d,d-a+b)} [v, v{-}1, 0] && (a-b \le d < a).
\end{aligned}$$

- If $2b > a$, then the transitions of the pruned boundary graph are

$$\begin{aligned}
[v, v-1, 0], [0, v-1, 0] &\xrightarrow{(d,d+a-b+1)} [0, 1-v, 0] && (0 \le d \le 2b - a - 2),\\
[v, v-1, 0], [0, v-1, 0] &\xrightarrow{(d,d+a-b+1)} [0, 1-v, v] && (2b - a - 1 \le d < b),\\
[0, 1-v, v], [0, 1-v, 0] &\xrightarrow{(d,d-a+b-1)} [0, v-1, 0] && (a - b < d < b),\\
[0, 1-v, v], [0, 1-v, 0] &\xrightarrow{(d,d-a+b-1)} [v, v-1, 0] && (b \le d \le a),\\
[0, v, v] &\xrightarrow{(d,d+a-b)} [0, 1-v, 0] && (0 \le d < 2b - a),\\
[0, v, v] &\xrightarrow{(d,d+a-b)} [0, 1-v, v] && (2b - a \le d < b),\\
[v, -v, 0] &\xrightarrow{(d,d-a+b)} [0, v-1, 0] && (a - b \le d < b),\\
[v, -v, 0] &\xrightarrow{(d,d-a+b)} [v, v-1, 0] && (b \le d < a).
\end{aligned}$$

To prove that these are exactly the transitions of the pruned boundary graph, note first that the states of the graph are of the form $[u, x, w]$ with $u, w \in \{0, v\}$ and $x \in \pm\{v, 1-v\}$. The only possibilities are thus $[0, v, v]$, $[0, 1-v, v]$, $[0, 1-v, 0]$ if $2v > 1$, $[v, v, v]$ if $2v < 1$, and their negatives $[v, -v, 0]$, $[v, v-1, 0]$, $[0, v-1, 0]$, and $[v, -v, v]$ respectively. Since $v = \frac{b}{\beta}$, we have $2v > 1$ if and only if $2b > a$. Moreover, the only possibility for $\frac{v+d}{\beta} \in \pm\{v, 1-v\}$ with $d \le a$ is that $\frac{v+d}{\beta} = 1 + \frac{d-a}{\beta} = 1-v$, i.e., $d = a - b$. Therefore, the above lists contain all possible transitions. Since we have an infinite path starting from each node, all given nodes correspond to the intersection of some tiles and are thus in the boundary graph.

From the description of the boundary graph, we see that the paths starting in a state $[u, x, w]$ depend only on x. Therefore, we can merge the states with same middle component and obtain the graph in Figure 3.7. Note that we have exactly $|N(\beta)|$ outgoing transitions from each state, which can be explained by the fact that the intersection $\mathcal{R}(x) \cap \mathcal{R}(x + \beta - \lfloor x + \beta \rfloor) \cap \mathbb{R} \times \{y\}$ consists of a singleton for each $x \in \mathbb{Z}[\beta]$, $y \in \overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])}$.

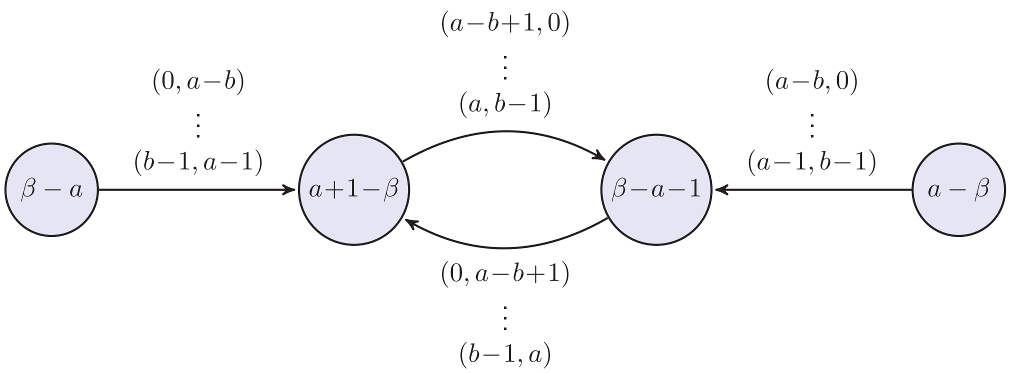


FIGURE 3.7. The pruned boundary graph (after merging the states with same middle component) for $\beta^2 = a\beta + b$, $a \ge b \ge 1$.

3.7.4. Proof of Theorem 3.6. Let $\beta^2 = a\beta + b$, $a \ge b \ge 1$, and $\beta' = -b\beta^{-1} = a - \beta$ be the Galois conjugate of β. By Theorem 3.5, we have

$$\begin{aligned}\gamma(\beta) \ge \inf\big(\{1\} \cup \big\{x \in \mathbb{Q} \cap [0, \beta{-}a) : \delta_\infty^c(-x) \in \pi_\infty^c\big(Z^c \setminus \mathcal{R}(0)\big)\big\} \\ \cup \big\{x \in \mathbb{Q} \cap [\beta{-}a, 1) : \delta_\infty^c(-x) \in \pi_\infty^c\big(Z^c \setminus \mathcal{R}(\beta{-}a)\big) - \delta_\infty^c(\beta{-}a)\big\}\big),\end{aligned} \tag{3.41}$$

with equality if $\overline{\delta_{\mathrm{f}}(\mathbb{Q})} = \mathbb{K}_{\mathrm{f}}$. By Lemma 1.30, the latter equality holds if $\gcd(a, b) = 1$. We have to show that the infimum is equal to $\max\{0, 1 - \frac{(b-1)b\beta}{\beta^2-b^2}\}$.

By (3.39) and its proof, $\pi_\infty^c(Z^c \setminus \mathcal{R}(0))$ is the union of two half-lines:

$$\begin{aligned}\pi_\infty^c\big(Z^c \setminus \mathcal{R}(0)\big) = \Big(-\infty, \max \pi_\infty^c\big(\mathcal{R}(0) \cap \mathcal{R}(\beta{-}a)\big)\Big) \\ \cup \Big(\min \pi_\infty^c\big(\mathcal{R}(0) \cap \mathcal{R}(a{+}1{-}\beta), \infty\big)\Big).\end{aligned}$$

The point in $\mathcal{R}(0) \cap \mathcal{R}(\beta{-}a)$ that realises $\max \pi_\infty^c(\mathcal{R}(0) \cap \mathcal{R}(\beta{-}a))$ is given by the following infinite walk in the pruned boundary graph starting from $\beta{-}a$: choose the transition to $a{+}1{-}\beta$ with maximal first digit $b{-}1$, then the transition to $\beta{-}a{-}1$ with minimal first digit $a{-}b{+}1$ (since we multiply it by an odd power of $\beta' < 0$), again the transition to $a{+}1{-}\beta$ with maximal first digit $b{-}1$, etc. This gives

$$\begin{aligned}\max \pi_\infty^c\big(\mathcal{R}(0) \cap \mathcal{R}(\beta{-}a)\big) &= \sum_{j=0}^{\infty} \big(b - 1 + (a - b + 1)\,\beta')\big)\,(\beta')^{2j} \\ &= \frac{b - 1 + (a - b + 1)\,\beta'}{1 - (\beta')^2} = \frac{(1 - b)\,\beta'}{1 - (\beta')^2} - 1 \\ &= \frac{(b - 1)b\,\beta}{\beta^2 - b^2} - 1,\end{aligned}$$

Note that $\delta_\infty^c(x) = x$ for $x \in \mathbb{Q}$. If $\frac{(b-1)b\,\beta}{\beta^2-b^2} \ge 1$, then we obtain that

$$\inf\{x \in \mathbb{Q} \cap [0, \beta{-}a) : \delta_\infty^c(-x) \in \pi_\infty^c(Z^c \setminus \mathcal{R}(0))\} = 0 = \max\big\{0, 1 - \tfrac{(b-1)b\,\beta}{\beta^2-b^2}\big\}.$$

Assume now that $\frac{(b-1)b\,\beta}{\beta^2-b^2} < 1$. Then, by similar calculations as above, we obtain that

$$\min \pi_\infty^c\big(\mathcal{R}(0) \cap \mathcal{R}(a{+}1{-}\beta)\big) = \frac{a - b + 1 + (b - 1)\,\beta'}{1 - (\beta')^2} = \beta\left(1 - \frac{(b - 1)\beta}{\beta^2 - b^2}\right) > 0.$$

Therefore, we have $\big[\frac{(b-1)b\,\beta}{\beta^2-b^2} - 1, 0\big] \times \overline{\delta_{\mathrm{f}}(\mathbb{Z}[\beta])} \subseteq \mathcal{R}(0)$, and

$$\inf\big\{x \in \mathbb{Q}\cap[0, 1) : \delta_\infty^c(-x) \in \pi_\infty^c\big(Z^c\setminus\mathcal{R}(0)\big)\big\} = 1 - \tfrac{(b-1)b\,\beta}{\beta^2-b^2} = \max\big\{0, 1 - \tfrac{(b-1)b\,\beta}{\beta^2-b^2}\big\}.$$

This concludes the case $0 < 1 - \frac{(b-1)b\,\beta}{\beta^2-b^2} < \beta - a$.

Assume now that $\frac{(b-1)b\,\beta}{\beta^2-b^2} \le a + 1 - \beta$. We see from the boundary graph that

$$\max \pi_\infty^c\big(\mathcal{R}(\beta{-}a) \cap \mathcal{R}(2\beta{-}\lfloor 2\beta\rfloor)\big) - \delta_\infty^c(\beta{-}a) = \max \pi_\infty^c\big(\mathcal{R}(0) \cap \mathcal{R}(\beta{-}a)\big)$$

and, since the smallest first digit in the outgoing transitions from $a{-}\beta$ is $a{-}b$,

$$\begin{aligned}\min \pi_\infty^c(\mathcal{R}(\beta{-}a) \cap \mathcal{R}(0)) - \delta_\infty^c(\beta{-}a) = \min \pi_\infty^c\big(\mathcal{R}(0) \cap \mathcal{R}(a{+}1{-}\beta)\big) - 1 \\ = \beta - \frac{(b - 1)\beta^2}{\beta^2 - b^2} - 1 \ge \beta - \frac{(a + 1 - \beta)\,\beta}{b} - 1 = \beta\left(1 - \frac{1}{b}\right) \ge 0.\end{aligned}$$

Similarly as above, we obtain that

$\inf\big\{x\in\mathbb{Q}\cap[\beta{-}a,1):\,\delta_\infty^c(-x)\in\pi_\infty^c\big(Z^c\setminus\mathcal{R}(\beta{-}a)\big)-\delta_\infty^c(\beta{-}a)\big\}=1-\frac{(b-1)b\,\beta}{\beta^2-b^2}$,

provided that $b\geq 2$. Finally, for $b=1$, we obtain Schmidt's equality $\gamma(\beta)=1$ [Sch80].

To conclude the proof of Theorem 3.6, we show that $\frac{(b-1)b\,\beta}{\beta^2-b^2}<1$ if and only if $(b-1)b<a$. Indeed, if $(b-1)b\geq a$, then

$$\beta^2-b^2-(b-1)b\,\beta\leq\beta^2-a\beta-b^2=b-b^2\leq -a<0,$$

and, if $(b-1)b<a$, then

$$\beta^2-b^2-(b-1)b\,\beta\geq\beta^2-(a-1)\,\beta-(a+b)=\beta-a>0.$$

3.7.5. Example of $\gamma(\beta)$. Let $\beta=\frac{3+\sqrt{17}}{2}$, i.e., $\beta^2=3\beta+2$. Then, by Theorem 3.6,

$$\gamma(\beta)=1-\frac{2\,\beta}{\beta^2-4}=\frac{1}{\beta+2}\approx 0,1798.$$

Figure 3.8 illustrates how to obtain $\gamma(\beta)$ as $\min\pi_\infty^c\big(-\mathcal{R}(0)\cap-\mathcal{R}(\beta{-}3)\big)$.

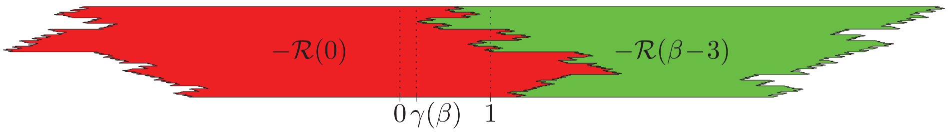


FIGURE 3.8. Visualization of $\gamma(\beta)$ for $\beta^2=3\beta+2$.

CHAPTER 4

Dynamics of reducible Pisot substitutions

In this chapter we set up a geometrical theory for the study of the dynamics of reducible Pisot substitutions based on Rauzy fractals generated by duals of higher dimensional extensions of substitutions. We obtain geometric representations of stepped surfaces and related polygonal tilings, self-replicating and periodic tilings made of Rauzy fractals for a family of reducible substitutions. We analyse the codings of a domain exchange defined on these fractal domains and we interpret them in a new combinatorial way. This chapter is based on [Min14].

General assumptions. In this chapter we will always assume that $\mathcal{A} = \{1, 2, \ldots, n\}$ and σ is a primitive unit Pisot substitution with Pisot root β such that $\deg(\beta) = d$.

4.1. Higher dimensional dual substitutions

We recall the definition and main properties of k-dimensional extensions of a substitution and their dual, first defined in [SAI01].

DEFINITION 4.1. We will denote by $(\mathbf{x}, a_1 \wedge \cdots \wedge a_k) \in \mathbb{Z}^n \times \bigwedge_{i=1}^k \mathcal{A}$ the *k-dimensional face* $\{\mathbf{x} + \sum_{i=1}^k t_i \mathbf{e}_{a_i} : t_i \in [0,1]\}$. We will assume the following:

- $(\mathbf{x}, a_1 \wedge \cdots \wedge a_k) = 0$ if $a_i = a_j$ for some i, j.
- Antisymmetry: $(\mathbf{x}, a_{\tau(1)} \wedge \cdots \wedge a_{\tau(k)}) = \operatorname{sgn}(\tau)(\mathbf{x}, a_1 \wedge \cdots \wedge a_k)$, where $\operatorname{sgn}(\tau)$ is the signature of the permutation τ.

Observe that this justifies the wedge product as choice of notation. The *type* of a face $(\mathbf{x}, a_1 \wedge \cdots \wedge a_k)$ will be $a_1 \wedge \cdots \wedge a_k$.

Let C_k be the free $\mathbb{Z}$-module with basis elements in $\mathbb{Z}^n \times \bigwedge_{i=1}^k \mathcal{A}$.

We will use multi-dimensional notation. Define $\underline{a} := a_1 \wedge \cdots \wedge a_k$ and write $\underline{a} \xrightarrow{p} \underline{b}$, $\underline{a} \xrightarrow{s} \underline{b}$ for $\sigma(a_i) = p_i b_i s_i$ for each a_i appearing in the wedge $\underline{a}$, b_i in $\underline{b}$, with emphasis on either the prefixes or suffixes; $\mathbf{l}(\underline{p})$ and $\mathbf{l}(\underline{s})$ will denote the abelianization of all prefixes p_i, suffixes s_i respectively.

DEFINITION 4.2. The *k-dimensional extension* of σ is the linear map on C_k

$$\mathbf{E}_k(\sigma)(\mathbf{x}, \underline{a}) = \sum_{\underline{a} \xrightarrow{\underline{p}} \underline{b}} (M_\sigma \mathbf{x} + \mathbf{l}(\underline{p}), \underline{b}). \tag{4.1}$$

Write $(\mathbf{x}, \underline{a})^*$ for the dual of the element $(\mathbf{x}, \underline{a}) \in C_k$. Let C_k^* be the free $\mathbb{Z}$-module generated by the basis elements $(\mathbf{x}, \underline{a})^*$. We are interested in the dual of $\mathbf{E}_k(\sigma)$.

PROPOSITION 4.3. *We have*

$$\mathbf{E}_k^*(\sigma)(\mathbf{x}, \underline{a})^* = \sum_{\underline{b} \xrightarrow{\underline{p}} \underline{a}} \left(M_\sigma^{-1}(\mathbf{x} - \mathbf{l}(\underline{p})), \underline{b}\right)^* \tag{4.2}$$

PROOF. Using the definition of dual of the linear map $\mathbf{E}_k(\sigma)$

$$\langle \mathbf{E}_k^*(\sigma)(\mathbf{x}, a_1\wedge\cdots\wedge a_k)^*, (\mathbf{y}, b_1\wedge\cdots\wedge b_k)\rangle = \langle (\mathbf{x}, a_1\wedge\cdots\wedge a_k)^*, \mathbf{E}_k(\sigma)(\mathbf{y}, b_1\wedge\cdots\wedge b_k)\rangle,$$

and using definition (4.1) we obtain that the scalar product is non-zero and equals $\mathrm{sgn}(\tau)$ only for those faces such that $M_\sigma \mathbf{y} + \mathbf{l}(p_1) + \cdots + \mathbf{l}(p_k) = \mathbf{x}$ and $b_1 \xrightarrow{p_1} a_{\tau(1)}, \ldots, b_k \xrightarrow{p_k} a_{\tau(k)}$ for some permutation τ. Thus, denoted $\mathbf{z} = M_\sigma^{-1}(\mathbf{x} - (\mathbf{l}(p_1) + \cdots + \mathbf{l}(p_k)))$, this means that $\mathrm{sgn}(\tau)(\mathbf{z}, b_1 \wedge \cdots \wedge b_k)^*$ appears in the image $\mathbf{E}_k^*(\sigma)(\mathbf{x}, \underline{a})^*$ and we can reorder it into $(\mathbf{z}, b_{\tau(1)} \wedge \cdots \wedge b_{\tau(k)})^*$ with $b_{\tau(1)} \xrightarrow{p_{\tau(1)}} a_1, \ldots, b_{\tau(k)} \xrightarrow{p_{\tau(k)}} a_k$ and rename. □

As observed in [SAI01] there is a boundary operator which associates with a k-dimensional face its boundary consisting in a union of oriented $(k-1)$-dimensional faces. A coboundary operator acting on duals of faces can be defined as well. An important property is that the boundary and coboundary operators commute with $\mathbf{E}_k(\sigma)$ and $\mathbf{E}_k^*(\sigma)$ respectively. All this can be done as in the classical simplicial homology and cohomology theory (see e.g. [Hat02]).

The dual maps are abstract objects formally defined on the dual basis, which has no geometric interpretation. We will interpret geometrically duals of faces of dimension k as faces of dimension $n-k$, in a Poincaré duality flavour (cf. [SAI01, AFHI11]).

DEFINITION 4.4. The map φ_k is defined by

$$\varphi_k : (\mathbf{x}, a_1 \wedge \cdots \wedge a_k)^* \mapsto (-1)^{a_1+\cdots+a_k}(\mathbf{x} + \mathbf{e}_{a_1} + \cdots + \mathbf{e}_{a_k}, b_1 \wedge \cdots \wedge b_{n-k})$$

where $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{n-k}\}$ form a partition of $\{1, 2, \ldots, n\}$ with $a_1 < \cdots < a_k$, $b_1 < \cdots < b_{n-k}$. If $\underline{a} = a_1 \wedge \cdots \wedge a_k$ we will write $\underline{a}^* = b_1 \wedge \cdots \wedge b_{n-k}$. We call $(\mathbf{x}, \underline{a}^*)$ the $(n-k)$-dimensional face *transverse* to $\underline{a}$.

It was shown in [SAI01] that this map commutes with the boundary and coboundary operators. Furthermore φ_k is invertible.

Now we can conjugate the dual maps by φ_k to obtain explicit geometric realizations.

DEFINITION 4.5. The *geometric dual map* $\mathbf{E}^k(\sigma)$ is defined as

$$\mathbf{E}^k(\sigma) \circ \varphi_{n-k} = \varphi_{n-k} \circ \mathbf{E}_{n-k}^*(\sigma).$$

Matrices. Notice that, since the geometric dual substitution $\mathbf{E}^k(\sigma)$ is conjugate to $\mathbf{E}_{n-k}^*(\sigma)$, it still depends on wedges of $n-k$ letters (see Equation (4.4) for $k = d-1$). Define the $(n-k) \times (n-k)$ matrix M_k associated with the operator $\mathbf{E}^k(\sigma)$ by

$$(M_k)_{\underline{a}\,\underline{b}} := |\mathbf{E}^k(\sigma)(\mathbf{0}, \underline{a})|_{\underline{b}} \tag{4.3}$$

where $|\cdot|_{\underline{b}}$ counts the number of times a face of type $\underline{b}$ occurs, taking orientation into account. The matrix M_k can be seen algebraically as the transpose of the $(n-k)$-th exterior power of M_σ, by computing all $(n-k) \times (n-k)$ minors. Its eigenvalues are the products of $n-k$ distinct eigenvalues of M_σ. Denote by $\rho(M_k)$ its leading eigenvalue (if it exists).

Prefix and suffix graph. We extend the definition of prefix and suffix graph (see Section 1.1) to this multi-dimensional setting. These are the graphs having the wedges $\underline{a}$ ordered lexicographically as nodes and weighted edges $\underline{a} \xrightarrow{p} \underline{b}$ (respectively $\underline{a} \xrightarrow{s} \underline{b}$) if $\sigma(\underline{a}) = \underline{pbs}$, where the weight of an edge is the sign of the permutation used to reorder $\underline{b}$. Finally we sum up the weights of identical edges.

We will use a simplified version of the suffix graph we have just defined. Precisely, we will consider the $\mathbf{E}^{n-k}(\sigma)$-*suffix graph* having the wedges $\underline{a}^*$ as nodes and weighted edges $\underline{a}^* \xrightarrow{s} \underline{b}^*$ where s is the concatenation of all suffixes appearing in $\underline{s}$ for $\sigma(\underline{a}) = \underline{pbs}$.

In general we have $n = \#\mathcal{A} \geq d = \deg(\beta)$ with strict inequality for reducible substitutions. We want to represent the action of the substitution geometrically on $\mathbb{K}^c_\beta$, thus it makes sense to consider the action of a dual substitution on $(d-1)$-dimensional faces. For this reason we will work with the geometric realization $\mathbf{E}^{d-1}(\sigma)$ conjugate to $\mathbf{E}^*_{n-d+1}(\sigma)$ by φ_{n-d+1}. An explicit formula for $\mathbf{E}^{d-1}(\sigma)$ reads as follows.

Proposition 4.6. *The following holds:*

$$\mathbf{E}^{d-1}(\sigma)(\mathbf{x}, \underline{a}^*) = \sum_{\underline{b} \xrightarrow{s} \underline{a}} (-1)^{\underline{a}+\underline{b}} \big(M_\sigma^{-1}(\mathbf{x} + \mathbf{l}(\underline{s})), \underline{b}^*\big) \tag{4.4}$$

where $(-1)^{\underline{a}}$ denotes $(-1)^{a_1+\cdots+a_{n-d+1}}$ with $a_i \in \underline{a}$ seen as numbers, $(\mathbf{x}, \underline{a}^*)$ is the $(d-1)$-dimensional face transverse to $\underline{a}$.

Proof. A face $(\mathbf{x}, \underline{a}^*)$ is sent by φ^{-1}_{n-d+1} to $(-1)^{\underline{a}}(\mathbf{x} - (\mathbf{e}_{a_1} + \cdots + \mathbf{e}_{a_{n-d+1}}), \underline{a})$. Applying $\mathbf{E}^*_{n-d+1}(\sigma)$ we get a sum of elements of the form

$$(-1)^{\underline{a}}\big(M_\sigma^{-1}(\mathbf{x} - (\mathbf{e}_{a_1} + \cdots + \mathbf{e}_{a_{n-d+1}}) - (\mathbf{l}(p_1) + \cdots + \mathbf{l}(p_{n-d+1}))), \underline{b}\big),$$

for $\sigma(b_i) = p_i a_i s_i$. Applying φ_{n-d+1} we have

$$(-1)^{\underline{a}+\underline{b}}\Big(M_\sigma^{-1}\big(\mathbf{x} - (\mathbf{l}(p_1 a_1) + \cdots + \mathbf{l}(p_{n-d+1} a_{n-d+1})) + M_\sigma(\mathbf{e}_{b_1} + \cdots + \mathbf{e}_{b_{n-d+1}})\big), \underline{b}^*\Big)$$

but, since $M_\sigma \mathbf{e}_{b_i} = \mathbf{l}(\sigma(b_i)) = \mathbf{l}(p_i a_i) + \mathbf{l}(s_i)$ we get the result. □

Similar formulas hold for general $\mathbf{E}^k(\sigma)$.

4.2. Stepped surfaces

We use some terminology as in [AFHI11], where they use a similar construction for a free group automorphism associated with a complex Pisot root. Recall $d = \deg(\beta)$.

Definition 4.7. A *patch* is a union of $(d-1)$-dimensional faces in $\mathbb{R}^n$. We will say that a patch is *in good position* with respect to $\mathbb{K}^c_\beta$ if the restriction of the projection of π_c to the patch is one-to-one. A *stepped surface* is a union of $(d-1)$-dimensional faces in $\mathbb{R}^n$ homeomorphic to $\mathbb{K}^c_\beta$ under π_c.

Notice that it is immediate that the projection by π_c of a stepped surface is a polygonal tiling of $\mathbb{K}^c_\beta$.

We look for a stepped surface invariant under (a power of) $\mathbf{E}^{d-1}(\sigma)$. An *m-seed patch* $\mathcal{U}$ is a union of $(d-1)$-dimensional faces based at $\mathbf{0}$ in good position such that $\mathcal{U} \subseteq \mathbf{E}^{d-1}(\sigma)^m(\mathcal{U})$, for some integer $m \geq 1$.

Given an m-seed patch $\mathcal{U}$ we will consider the following potential candidate for a stepped surface:

$$\Gamma_{\mathcal{U}} := \bigcup_{k \geq 0} \mathbf{E}^{d-1}(\sigma)^{mk}(\mathcal{U}). \tag{4.5}$$

The next property will be crucial in the sequel.

DEFINITION 4.8. We say that σ is *regular* if the sequence $(\mathbf{E}^{d-1}(\sigma)^{mk}(\mathcal{U}))_{k\geq 0}$ is in good position, for each m-seed patch $\mathcal{U}$.

Observe that regularity implies that $\mathbf{E}^{d-1}(\sigma)(\mathbf{0}, \underline{a})$ is in good position for each $\underline{a}$. We conjecture that the latter is enough to deduce that $\Gamma_{\mathcal{U}}$ is in good position, but problems could arise in proving that $\mathbf{E}^{d-1}(\sigma)((\mathbf{x}, \underline{a}) \cup (\mathbf{y}, \underline{b}))$ is in good position for $(\mathbf{x}, \underline{a}) \cup (\mathbf{y}, \underline{b})$ in $\Gamma_{\mathcal{U}}$ in good position.

We do not know whether $\pi_c(\Gamma_{\mathcal{U}})$ covers the entire representation space $\mathbb{K}_\beta^c$. This motivates the next definition.

DEFINITION 4.9. We say that σ satisfies the *geometric finiteness property for* $\mathcal{U}$ if $\pi_c(\Gamma_{\mathcal{U}})$ is a covering of $\mathbb{K}_\beta^c$.

We can give a sufficient condition so that the geometric finiteness property is satisfied.

In order to do this we will consider powers of the matrices M_{d-1} and M_{d-2} associated with $\mathbf{E}^{d-1}(\sigma)$ and $\mathbf{E}^{d-2}(\sigma)$ respectively, defined in (4.3), which describe respectively the growth of patches of $(d-1)$ and $(d-2)$-dimensional faces. From now on we will suppose that these matrices are primitive. The main point is that these matrices may have negative entries. Thus cancellation may occur by taking powers of M_{d-1} or M_{d-2}. Now there are two cases:

- Cancellation is "good", in the sense that two faces based at the same point and with the same type but opposite orientations cancel.
- Cancellation is "bad", in the sense that two faces with the same type and opposite orientations are cancelled but they should not since they have different base points, but this happens because the effect of abelianisation does not recognize it.

Good cancellation happens for example for the Tribonacci substitution. But in general we are in the bad cancellation case, as it happens for the Hokkaido substitution (see Section 4.5).

For this reason we will consider the non-negative matrices M'_{d-1} and M'_{d-2}, defined by

$$(M'_{d-1})_{\underline{a}\,\underline{b}} = |(M_{d-1})_{\underline{a}\,\underline{b}}|, \qquad (M'_{d-2})_{\underline{a}\,\underline{b}} = |(M_{d-2})_{\underline{a}\,\underline{b}}|. \tag{4.6}$$

They consider everything under the same orientation and their use prevents that bad cancellation occurs. The drawback is that good cancellation is ruined: some coincident faces with opposite orientation which should cancel are anyway counted with positive multiplicities. However, if the growth of the matrix M'_{d-1} is greater than the growth of the matrix M'_{d-2}, which describes the boundary of $(d-1)$-dimensional patches, we can say that the geometric finiteness condition holds. We make it more precise in the next proposition.

PROPOSITION 4.10. *If $\rho(M'_{d-1}) > \rho(M'_{d-2})$ then we can cover disks of arbitrarily large radius by iterating $\mathbf{E}^{d-1}(\sigma)$ on any non-empty patch and projecting.*

Proof. Let P_k be the k-th iterate by $\mathbf{E}^{d-1}(\sigma)$ on a non-empty patch P. The ratio $\#P_{k+1}/\#P_k$ will be greater than $\rho(M'_{d-1}) - \epsilon$, $\forall\, \varepsilon > 0$. The boundary of the patch P_{k+1} is $\partial P_{k+1} = \partial\mathbf{E}^{d-1}(P_k) = \mathbf{E}^{d-2}(\partial P_k)$. Thus the ratio of faces touching the boundary of P_{k+1} to those touching the boundary of P_k will be less than $\rho(M'_{d-2}) + \varepsilon$. Hence

$$\frac{\#\partial P_k}{\#P_k} = O\left(\frac{(\rho(M'_{d-2}) + \varepsilon)^k}{(\rho(M'_{d-1}) - \varepsilon)^k}\right) \tag{4.7}$$

which approaches 0 for $k \to \infty$ if $\rho(M'_{d-1}) > \rho(M'_{d-2})$. We deduce that we can cover disks of arbitrarily large radius by iterating $\mathbf{E}^{d-1}(\sigma)$ on the non-empty patch P. Otherwise any fixed face in P_k would stay at bounded distance from the boundary ∂P_k for each k, and the ratio (4.7) would be bounded by some positive constant. □

The aim of this section is to obtain stepped surfaces for reducible substitutions. Now we have all the necessary ingredients.

Theorem 4.11. *Let σ be a regular substitution which satisfies the geometric finiteness property for the m-seed patch $\mathcal{U}$. Then $\Gamma_{\mathcal{U}}$ is a stepped surface invariant under the substitution rule associated with $\mathbf{E}^{d-1}(\sigma)^m$. Furthermore $\pi(\Gamma_{\mathcal{U}})$ stays within bounded distance of $\mathbb{K}^c_\beta$.*

Proof. The primitivity of M_{d-1} assures that there exists an m-seed patch. By the geometric finiteness property $\pi_c(\Gamma_{\mathcal{U}})$ is a covering of $\mathbb{K}^c_\beta$ and indeed a polygonal tiling since the substitution is regular. Furthermore it is by definition invariant under $\mathbf{E}^{d-1}(\sigma)^m$. $\pi(\Gamma_{\mathcal{U}})$ stays within bounded distance of $\mathbb{K}^c_\beta$ since the elements of $\pi_e(\Gamma_{\mathcal{U}})$ are all of the form $\sum_{i\geq 1} v_{\underline{s}_i}\beta^{-i}$, with the $v_{\underline{s}_i} = \pi_e(\mathbf{l}(\underline{s}_i))$ in a finite set, and $\beta^{-1} < 1$. □

We can consider the stepped surface also as a set of coloured points. In this case we denote it by $\Gamma^\bullet_{\mathcal{U}}$.

Corollary 4.12. *The elements of $\pi_c(\Gamma^\bullet_{\mathcal{U}})$ form a Delone set.*

Proof. It is an easy consequence of the fact that $\pi_c(\Gamma)$ forms a polygonal tiling of $\mathbb{K}^c_\beta$. □

Remark 4.13. Similarly as in the irreducible case an abstract stepped surface seen as set of nearest coloured points above $\mathbb{K}^c_\beta$ is defined in [EIR06] as

$$\mathcal{S}_1 = \{(\pi(\mathbf{x}), a) \in \pi(\mathbb{Z}^n) \times \mathcal{A} : \pi_e(\mathbf{x}) \in [0, v_a)\}. \tag{4.8}$$

It was shown that this set is invariant under the operator $\mathbf{E}^*_1(\sigma)$ conjugate to $\mathbf{E}^{n-1}(\sigma)$. Its projection into $\mathbb{K}^c_\beta$ gives a quasi-periodic Delone set which acts naturally as translation set for a self-replicating multiple tiling of $\mathbb{K}^c_\beta$. However they were not able to get a geometric representation for this stepped surface.

The set $\Gamma_{\mathcal{U}}$ depends strongly from the initial patch $\mathcal{U}$ we choose. It seems difficult to characterize this set in a similar fashion as (4.8). Computer experiments suggest that $\pi(\Gamma^\bullet_{\mathcal{U}}) \subset \mathcal{S}_1$. Iterations of $\mathbf{E}^{d-1}(\sigma)$ on the initial patch $\mathcal{U}$ select some of the points of $\mathcal{S}_1$ and generate a Delone set, which, as we will see in the next section, if the geometric finiteness property holds, provides a translation set for a tiling made of Rauzy fractals.

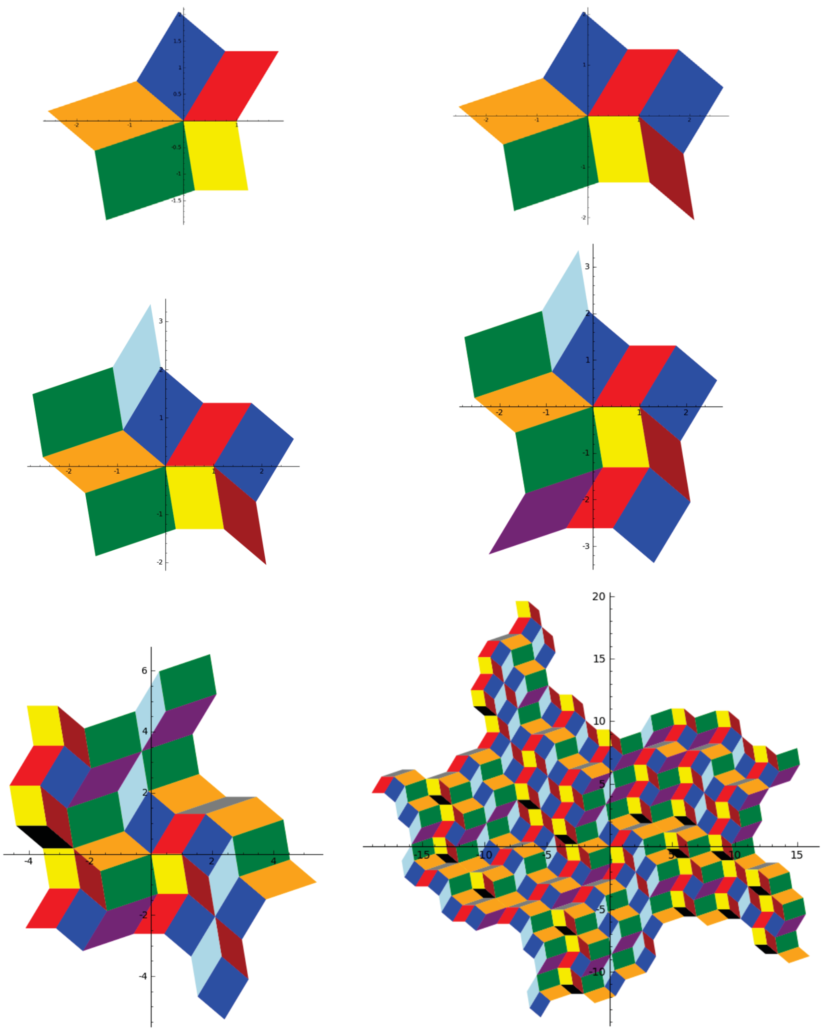


FIGURE 4.1. $\pi_c(\mathbf{E}^2(\sigma)^k(\mathcal{U}))$ for $k = 0, 1, 2, 3, 7, 15$ and $\mathcal{U} = \{(\mathbf{0}, 1 \wedge 3), (\mathbf{0}, 1 \wedge 4), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 2 \wedge 5), (\mathbf{0}, 3 \wedge 5)\}$.

4.3. Rauzy fractals and aperiodic tilings

From now on we will assume that σ is a regular Pisot substitution.

PROPOSITION 4.14. *For any* $(\mathbf{x}, \underline{a})$, *the sequence of sets* $\beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathbf{x}, \underline{a}))$ *converges in the Hausdorff metric.*

PROOF. The Hausdorff distance $d_H((\mathbf{x}, \underline{a}), \beta \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)(\mathbf{x}, \underline{a})))$ is uniformly bounded. The action of β in $\mathbb{K}_\beta^c$ is a contraction and we obtain that the Hausdorff distance between two successive sets in the sequence $(\beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathbf{x}, \underline{a})))_{k\in\mathbb{N}}$ decreases geometrically fast. Hence the sequence $(\beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathbf{x}, \underline{a})))_{k\in\mathbb{N}}$ is a Cauchy sequence which converges to a unique compact set in the Hausdorff topology. □

DEFINITION 4.15. The *Rauzy fractals* are defined as

$$\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}) = \lim_{k\to\infty} \beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathbf{x}, \underline{a})),$$

where the limit is taken with respect to the Hausdorff metric.

Recall that $g(x)$ is the neutral polynomial of σ. We introduce the condition

(N) $g(x)$ has only roots of modulus one.

REMARK 4.16. Condition (N) is related to *homological Pisot substitutions* defined in [BBJS12]. Indeed, if σ is homological Pisot with characteristic polynomial $f(x)g(x)$, where $g(x)$ is the neutral polynomial, then all roots of $g(x)$ are zero or roots of unity.

LEMMA 4.17. *Suppose M_{d-1} is primitive. If (N) holds then the Perron eigenvalue of M_{d-1} is β.*

PROOF. Recall that the eigenvalues of M_{d-1} are the products of $n-d+1$ distinct eigenvalues of M_σ. Thus the modulus of the Perron eigenvalue of M_{d-1} is $|\beta \prod_i \zeta_i| = \beta$, where the ζ_i are the $n-d$ roots of the unimodular neutral polynomial $g(x)$. □

PROPOSITION 4.18. *We have the set equations*

$$\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}) = \bigcup_{(\mathbf{y},\underline{b})\in\mathbf{E}^{d-1}(\sigma)(\mathbf{x},\underline{a})} \beta \cdot \big(\mathcal{R}(\underline{b}) + \pi_c(\mathbf{y})\big). \tag{4.9}$$

Furthermore if (N) holds the union on the right-hand side is measure disjoint.

PROOF. The set equations follow easily by definition of Rauzy fractal. For the measures the following holds

$$\beta\ \mu_c(\mathcal{R}(\underline{a})) \le \sum_{\underline{b}} m_{\underline{a}\,\underline{b}}\ \mu_c(\mathcal{R}(\underline{b}))$$

where $m_{\underline{a}\,\underline{b}}$ denote the entries of M_{d-1}. We get the equality since by Lemma 4.17 the Perron eigenvalue of M_{d-1} is β. □

It follows from the above proposition that the vector of measures $(\mu_c(\mathcal{R}(\underline{a})))_{\underline{a}}$ is a (non-zero) Perron eigenvector of M_{d-1}.

Assume for the rest of the section that (N) holds. Lemma 4.17 allows us to use the result [LW03, Theorem 5.5] on substitution Delone sets to get nice properties for our Rauzy fractals (see also [EIR06, Section 6]).

PROPOSITION 4.19. *The Rauzy fractals have the following properties:*

(i) they are compact sets with non-zero measure.
(ii) they are the closure of their interior.
(iii) their fractal boundary has zero measure.

We are interested in aperiodic tilings made of Rauzy fractals. In the following theorem we show that the geometric finiteness condition is sufficient to obtain aperiodic tilings.

THEOREM 4.20. *Given a seed patch $\mathcal{U}$, if the geometric finiteness property for $\mathcal{U}$ holds then the collection $\mathcal{C}_\mathcal{U} = \{\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}) : (\mathbf{x}, \underline{a}) \in \Gamma_\mathcal{U}\}$ is a self-replicating tiling of $\mathbb{K}_\beta^c$.*

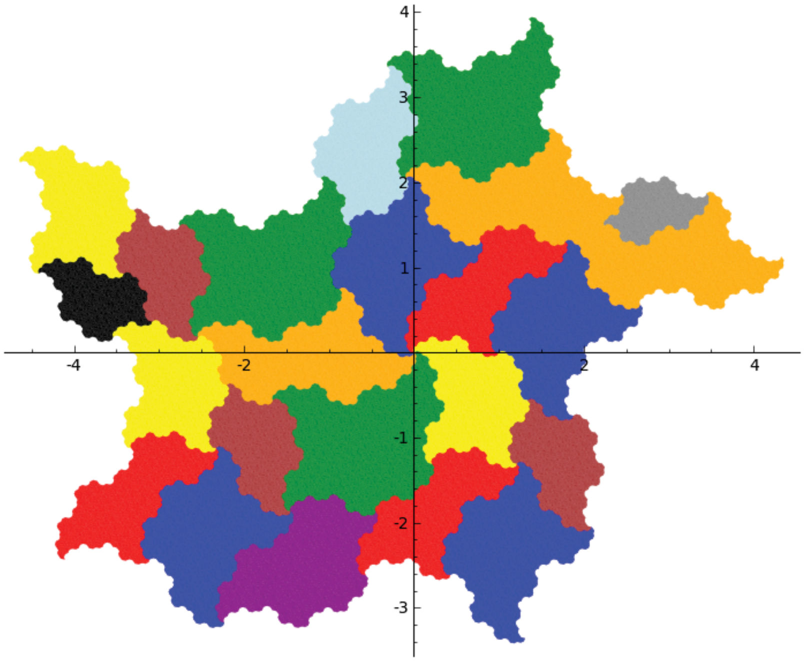


FIGURE 4.2. A patch of the self-replicating tiling generated by the patch $\mathcal{U}$.

PROOF. By the geometric finiteness property $\pi_c(\Gamma_{\mathcal{U}})$ is a covering and by definition of $\Gamma_{\mathcal{U}}$ and its $\mathbf{E}^{d-1}(\sigma)$-invariance we have that $\mathcal{C}_{\mathcal{U}}$ is a covering, i.e.,

$$\mathbb{K}_\beta^c = \bigcup_{k\geq 0} \bigcup_{(\mathbf{x},\underline{a})\in\mathbf{E}^{d-1}(\sigma)^k(\mathcal{U})} \mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}).$$

By Proposition 4.18, iterating the set equations (4.9) we get that the union of the $\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x})$ such that $(\mathbf{x},\underline{a}) \in \mathbf{E}^{d-1}(\sigma)^k(\mathcal{U})$ is measure disjoint. This implies that $\mathcal{C}_{\mathcal{U}}$ is a tiling. □

4.4. Periodic tilings

One of the main novelties is that we obtain natural periodic tilings by our Rauzy fractals starting from periodic polygonal tilings.

DEFINITION 4.21. A *periodic patch* is a polygonal patch $\mathcal{P}$ which, translated by a set of points $\Lambda_{\mathcal{P}} \subset \mathbb{Z}^n$ such that $\pi_c(\Lambda_{\mathcal{P}})$ is a lattice, forms a stepped surface. In this case we will call the latter a *periodic stepped surface.*

Examples of periodic stepped surfaces are:

(1) A single $(d-1)$-dimensional face $(\mathbf{0}, a_1 \wedge \cdots \wedge a_{d-1})$ together with $\Lambda_{\mathcal{P}} = \mathbf{e}_{a_1}\mathbb{Z} + \cdots + \mathbf{e}_{a_{d-1}}\mathbb{Z}$.
(2) A *touching pair*, that is, a patch of two faces in good position which differ only in one letter: $(\mathbf{0}, b \wedge a_2 \wedge \cdots \wedge a_{d-1})$, $(\mathbf{0}, c \wedge a_2 \wedge \cdots \wedge a_{d-1})$, $b \neq c$. The associated set is $\Lambda_{\mathcal{P}} = (\mathbf{e}_b - \mathbf{e}_c)\mathbb{Z} + \sum_{i=2}^{d-1} \mathbf{e}_{a_i}\mathbb{Z}$.

(3) A *d-touching* patch, that is, a patch of $d = \deg(\beta)$ faces which are touching in pairs $\sum_{k=1}^{d}(\mathbf{0}, a_1 \wedge \cdots \wedge \widehat{a}_k \wedge \cdots \wedge a_d)$, where $\widehat{a}$ denotes that a does not appear. The associated set is $\Lambda_{\mathcal{P}} = \sum_{i=2}^{d}(\mathbf{e}_{a_1} - \mathbf{e}_{a_i})\mathbb{Z}$.

We will need the following.

Assumption: the image by $\mathbf{E}^{d-1}(\sigma)$ of a stepped surface is a stepped surface.

We strongly believe that this assumption is true and that can be proven with similar techniques as in [Fer06, ABFJ07, BF11].

Proposition 4.22. *Let $\mathcal{P} + \Lambda_{\mathcal{P}}$ be a periodic stepped surface. Then $\beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathcal{P})) + \pi_c(\Lambda_{\mathcal{P}})$ provides a polygonal periodic tiling of $\mathbb{K}_\beta^c$, for every $k \geq 0$.*

Proof. Since by assumption the projected image by $\mathbf{E}^{d-1}(\sigma)$ of $\mathcal{P} + \Lambda_{\mathcal{P}}$ is again a polygonal tiling, we have that

$$\mu_c\big(\beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathbf{x}, \underline{a})) \cap \beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathbf{y}, \underline{b})\big) = 0$$

for any two faces $(\mathbf{x}, \underline{a})$, $(\mathbf{y}, \underline{b}) \in \mathcal{P} + \Lambda_{\mathcal{P}}$. Furthermore $\mu_c(\beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathbf{x}, \underline{a}))) = \mu_c(\pi_c(\mathbf{x}, \underline{a}))$, since $(\mu_c(\pi_c(\mathbf{0}, \underline{a})))_{\underline{a}}$ is a Perron-Frobenius eigenvector of M_{d-1} associated with β. Thus $\mathcal{P}_k + \Lambda_{\mathcal{P}}$ is a periodic tiling. □

Now we replace $\pi_c(\mathcal{P})$ with the Rauzy fractals. Denote $\mathcal{R}_{\mathcal{P}} = \bigcup_{(\mathbf{0}, \underline{a}) \in \mathcal{P}} \mathcal{R}(\underline{a})$.

Corollary 4.23. *$\mathcal{R}_{\mathcal{P}} + \pi_c(\Lambda_{\mathcal{P}})$ is a periodic covering of $\mathbb{K}_\beta^c$.*

Proof. It follows from Proposition 4.22 and from the fact that $\mathcal{R}_{\mathcal{P}}$ is the Hausdorff limit of the approximations $\beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\mathcal{P}))$. □

Let $\mathcal{R}_k(\underline{a}) := \beta^k \cdot \pi_c(\mathbf{E}^{d-1}(\sigma)^k(\underline{a}))$.

Theorem 4.24. *Let $\mathcal{P} + \Lambda_{\mathcal{P}}$ a periodic stepped surface. Then $\mathcal{R}_{\mathcal{P}} + \pi_c(\Lambda_{\mathcal{P}})$ forms a periodic tiling if and only if $\lim_{k \to \infty} \partial \mathcal{R}_k(\underline{a}) = \partial \mathcal{R}(\underline{a})$, for all $\underline{a}$.*

Proof. Assume $\lim_{k \to \infty} \partial \mathcal{R}_k(\underline{a}) = \partial \mathcal{R}(\underline{a})$. Then, since $\mathcal{R}_k(\underline{a}) \to \mathcal{R}(\underline{a})$, we have that for any $\varepsilon > 0$ there exists $k = k(\varepsilon)$ such that

$$d_H(\mathcal{R}(\underline{a}), \mathcal{R}_k(\underline{a})) < \varepsilon \quad \text{and} \quad d_H(\partial \mathcal{R}(\underline{a}), \partial \mathcal{R}_k(\underline{a})) < \varepsilon.$$

By the former, for any $z \in \mathcal{R}(\underline{a}) \setminus \mathcal{R}_k(\underline{a})$ there exists $z' \in B(z, \varepsilon) \cap \mathcal{R}_k(\underline{a})$, which implies that the line segment from z to z' must intersect $\partial \mathcal{R}_k(\underline{a})$. Hence there exists $z'' \in \partial \mathcal{R}_k(\underline{a})$ such that $|z - z''| < \varepsilon$, and

$$\mathcal{R}(\underline{a}) \subseteq \mathcal{R}_k(\underline{a}) \cup [\partial \mathcal{R}_k(\underline{a})]_\varepsilon,$$

where $[X]_\varepsilon = \{x : |x - y| < \varepsilon \text{ for some } y \in X\}$. Since the approximations $\mathcal{R}_k(\underline{a})$ have for every k the same measure as the projected faces $\pi_c(\mathbf{0}, \underline{a})$ and $\mathcal{R}_{\mathcal{P}} + \Lambda_{\mathcal{P}}$ is a covering we have $\mu_c(\mathcal{R}_{\mathcal{P}})/\mu_c(\mathcal{P}_k) \geq 1$. Thus we get equality if $\lim_{\varepsilon \to 0} \mu_c([\partial \mathcal{R}_k(\underline{a})]_\varepsilon) = 0$. But the inequality $d_H(\partial \mathcal{R}(\underline{a}), \partial \mathcal{R}_k(\underline{a})) < \varepsilon$ implies that $[\partial \mathcal{R}_k(\underline{a})]_\varepsilon \subseteq [\partial \mathcal{R}(\underline{a})]_{2\varepsilon}$ and $\lim_{\varepsilon \to 0} \mu_c([\partial \mathcal{R}(\underline{a})]_{2\varepsilon}) = 0$ since $\partial \mathcal{R}(\underline{a})$ has measure zero.

Suppose $\mathcal{R}_{\mathcal{P}} + \Lambda_{\mathcal{P}}$ is a tiling. Then $\beta^{-k} \cdot \mathcal{R}_{\mathcal{P}}$ and $\beta^{-k} \cdot \bigcup_{(\mathbf{0}, \underline{a}) \in \mathcal{P}} \mathcal{R}_k(\underline{a})$ tile both $\mathbb{K}_\beta^c$ modulo $\beta^{-k} \cdot \Lambda_{\mathcal{P}}$. Let

$$C = \max_{\underline{a}}\{\operatorname{diam}(\mathcal{R}(\underline{a}))\} + \max_{\underline{a}}\{\operatorname{diam}(\pi_c(\mathbf{0}, \underline{a}))\}.$$

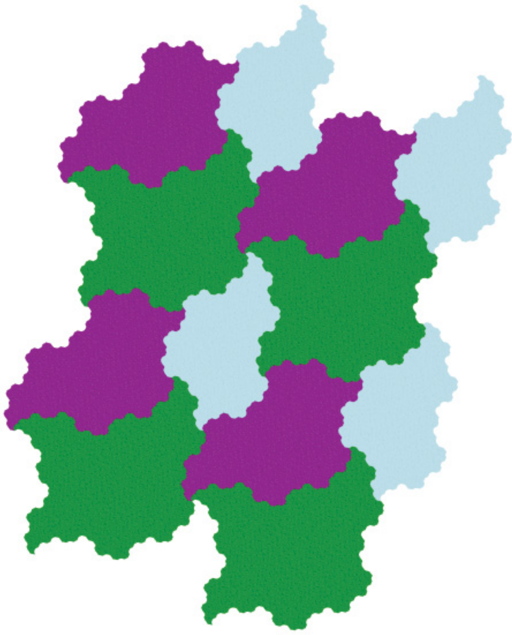


Figure 4.3. A patch of the periodic tiling $\mathcal{R}_{\mathcal{P}}+\pi_c(\Lambda_{\mathcal{P}})$, for $\mathcal{P}=\{(\mathbf{0},2\wedge 3),(\mathbf{0},2\wedge 4),(\mathbf{0},3\wedge 4)\}$ and $\Lambda_{\mathcal{P}}=(\mathbf{e}_4-\mathbf{e}_2)\mathbb{Z}+(\mathbf{e}_4-\mathbf{e}_3)\mathbb{Z}$.

If $B(z,R)\subset\beta^{-k}\cdot\mathcal{R}_{\mathcal{P}}$ then $B(z,R-C)\subset\beta^{-k}\cdot\bigcup_{(\mathbf{0},\underline{a})\in\mathcal{P}}\mathcal{R}_k(\underline{a})$, and $B(z,R)\subset\beta^{-k}\cdot\bigcup_{(\mathbf{0},\underline{a})\in\mathcal{P}}\mathcal{R}_k(\underline{a})$ implies $B(z,R-C)\subset\beta^{-k}\cdot\mathcal{R}_{\mathcal{P}}$. Hence

$$d_H(\partial(\beta^{-k}\cdot\mathcal{R}(\underline{a})),\partial(\beta^{-k}\cdot\mathcal{R}_k(\underline{a}))<2C,$$

and the result follows. □

Proposition 4.25. *Let $\mathcal{P}$ be a periodic seed patch. If the geometric finiteness property for $\mathcal{P}$ holds, then $\mathcal{R}_{\mathcal{P}}+\pi_c(\Lambda_{\mathcal{P}})$ is a tiling.*

Proof. If the geometric finiteness property for $\mathcal{P}$ holds then $\mathcal{C}_{\mathcal{P}}=\{\mathcal{R}(\underline{a})+\pi_c(\mathbf{x}):(\mathbf{x},\underline{a})\in\Gamma_{\mathcal{P}}\}$ and $\pi_c(\Gamma_{\mathcal{P}})$ are tilings with the same translation set. This implies that $\mu_c(\mathcal{R}(\underline{a}))=\mu_c(\pi_c(\mathbf{0},\underline{a}))$, for all $\underline{a}$, in particular for those appearing in the patch $\mathcal{P}$, and we know $\mathcal{R}_{\mathcal{P}}+\Lambda_{\mathcal{P}}$ is a covering by Corollary 4.23. Therefore $\mathcal{R}_{\mathcal{P}}+\Lambda_{\mathcal{P}}$ is a tiling. □

4.4.1. Domain exchange. We introduce an important combinatorial condition on the substitutions.

Definition 4.26. Given a non-overlapping patch $\mathcal{P}$, we say that the substitution σ satisfies the $\mathcal{P}$-*strong coincidence condition* if for every pair $(\mathbf{0},\underline{b}_1)$, $(\mathbf{0},\underline{b}_2)\in\mathcal{P}$, there exist $k\in\mathbb{N}$ and $\underline{a}$ such that $\sigma^k(\underline{b}_1)=\underline{p}_1\underline{a}\underline{s}_1$ and $\sigma^k(\underline{b}_2)=\underline{p}_2\underline{a}\underline{s}_2$ with $\mathbf{l}(\underline{s}_1)=\mathbf{l}(\underline{s}_2)$.

Remark 4.27. Notice that in the irreducible case we can take the non-overlapping patch $\mathcal{P}=\bigcup_{a\in\mathcal{A}}(\mathbf{0},a^*)$ and the $\mathcal{P}$-strong coincidence condition coincides with the *strong coincidence condition* (see Definition 1.3).

Proposition 4.28. *If σ satisfies (N) and the $\mathcal{P}$-strong coincidence condition, for $\mathcal{P}$ non-overlapping patch, then the subtiles $\mathcal{R}(\underline{a})$, $\underline{a} \in \mathcal{P}$, are measure disjoint.*

Proof. For every $\underline{b}_1, \underline{b}_2 \in \mathcal{P}$ there exist $k \in \mathbb{N}$ and $\underline{a}$ such that $\sigma^k(\underline{b}_1) = \underline{p}_1 \underline{a} \underline{s}_1$ and $\sigma^k(\underline{b}_2) = \underline{p}_2 \underline{a} \underline{s}_2$ with $\mathbf{l}(\underline{s}_1) = \mathbf{l}(\underline{s}_2)$. Thus $\beta^k \cdot \mathcal{R}(\underline{b}_1) + \pi_c(\mathbf{l}(\underline{s}_1))$ and $\beta^k \cdot \mathcal{R}(\underline{b}_2) + \pi_c(\mathbf{l}(\underline{s}_2))$ both appear in the k-fold iteration of the set equations of Proposition 4.18 for $\mathcal{R}(\underline{a})$, and furthermore are measure disjoint. □

Recall the definition of a d-touching patch given at the beginning of Section 4.4.

Definition 4.29. Let $\mathcal{P}$ be a d-touching patch and let $\mathcal{A}_{\mathcal{P}}$ be the alphabet of the patch, consisting of the single letters appearing in the wedges of the patch $\mathcal{P}$. If the $\mathcal{P}$-strong coincidence condition holds, the *domain exchange* on $\mathcal{R}_{\mathcal{P}}$ is defined as

$$E_{\mathcal{P}} : \mathcal{R}(\underline{a}) \mapsto \mathcal{R}(\underline{a}) + \delta_c(v_\ell), \tag{4.10}$$

for $\ell \in \mathcal{A}_{\mathcal{P}} \setminus \{a_1, \dots, a_{d-1}\}$, $\underline{a} = a_1 \wedge \cdots \wedge a_{d-1}$.

Let $\mathcal{P}$ be a d-touching patch with associated set $\Lambda_{\mathcal{P}}$ and assume that σ satisfies the $\mathcal{P}$-strong coincidence condition. Then it is clear that, if $\mathcal{R}_{\mathcal{P}} + \pi_c(\Lambda_{\mathcal{P}})$ is a periodic tiling, then $E_{\mathcal{P}}$ is a translation on the $(d-1)$-dimensional torus $\mathbb{K}_\beta^c / \pi_c(\Lambda_{\mathcal{P}})$.

We are interested now in codings of the domain exchange $E_{\mathcal{P}}$ with respect to the natural partition given by $\mathcal{R}_{\mathcal{P}}$. The question is whether we can generalize the techniques used in the irreducible case and prove the measure-theoretic conjugations of the diagram

$$\begin{array}{ccccc} \Omega & \longrightarrow & \mathcal{R}_{\mathcal{P}} & \longrightarrow & \mathbb{K}_\beta^c/\Lambda_{\mathcal{P}} \\ {\scriptstyle S}\downarrow & & {\scriptstyle E_{\mathcal{P}}}\downarrow & & {\scriptstyle E_{\mathcal{P}}}\downarrow \\ \Omega & \longrightarrow & \mathcal{R}_{\mathcal{P}} & \longrightarrow & \mathbb{K}_\beta^c/\Lambda_{\mathcal{P}} \end{array}$$

where Ω is a symbolic dynamical systems which codes the orbits of $(\mathcal{R}_{\mathcal{P}}, E_{\mathcal{P}})$. We would like to investigate further and know what the connections with the original substitution dynamical system (X_σ, S) are. We will provide a complete answer to these questions for a family of reducible Pisot substitutions in the next section.

4.5. A family of regular substitutions

We consider the family of substitutions

$$\sigma_t : \ 1 \mapsto 1^{t+1}2, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 1^t 5, \ 5 \mapsto 1$$

with associated polynomials

$$f(x)g(x) = (x^3 - tx^2 - (t+1)x - 1)(x^2 - x + 1), \quad t \in \mathbb{N}_0.$$

Note that σ_0 is the substitution associated with the minimal Pisot number. Being $\#\mathcal{A} = 5$ and $\deg(\beta) = 3$ we will consider the geometric dual substitution $\mathbf{E}^2(\sigma)$ conjugate to $\mathbf{E}_3^*(\sigma)$. We say that a face $(\mathbf{x}, a \wedge b)$ is positively oriented if $a < b$.

For σ_0 the dual map is given explicitly by

$$\begin{aligned}
\mathbf{E}^2(\sigma_0) : (\mathbf{0}, 4\wedge 5) &\mapsto (\mathbf{0}, 3\wedge 4)\\
(\mathbf{0}, 3\wedge 5) &\mapsto (\mathbf{0}, 2\wedge 4)\\
(\mathbf{0}, 3\wedge 4) &\mapsto (\mathbf{0}, 2\wedge 3)\\
(\mathbf{0}, 2\wedge 5) &\mapsto (\mathbf{0}, 1\wedge 4) \cup (M_\sigma^{-1}\mathbf{e}_2, 4\wedge 5)\\
(\mathbf{0}, 2\wedge 4) &\mapsto (\mathbf{0}, 1\wedge 3) \cup (M_\sigma^{-1}\mathbf{e}_2, 3\wedge 5)\\
(\mathbf{0}, 2\wedge 3) &\mapsto (\mathbf{0}, 1\wedge 2) \cup (M_\sigma^{-1}\mathbf{e}_2, 2\wedge 5)\\
(\mathbf{0}, 1\wedge 5) &\mapsto -(\mathbf{0}, 4\wedge 5)\\
(\mathbf{0}, 1\wedge 4) &\mapsto -(\mathbf{0}, 3\wedge 5)\\
(\mathbf{0}, 1\wedge 3) &\mapsto -(\mathbf{0}, 2\wedge 5)\\
(\mathbf{0}, 1\wedge 2) &\mapsto -(\mathbf{0}, 1\wedge 5)
\end{aligned}$$

In Figure 4.4 the $\mathbf{E}^2(\sigma_t)$-suffix graph is depicted. Observe that this graph with reversed edges describes the images of every face by $\mathbf{E}^2(\sigma_t)$.

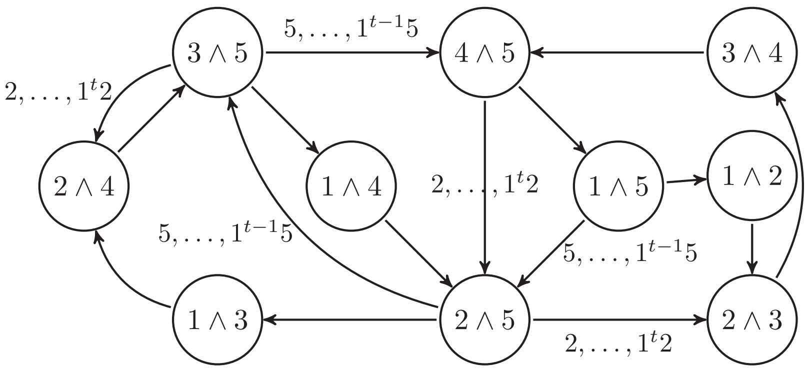


FIGURE 4.4. The $\mathbf{E}^2(\sigma_t)$-suffix graph. Every unlabelled edge has the empty suffix as label, while the edges labelled by $5, \dots, 1^{t-1}5$ exist only for $t \geq 1$.

We say that a patch of faces $(F_k)_{k=0}^m$ forms a *chain* if F_k and F_{k+1} have an edge in common and are in good position.

PROPOSITION 4.30. *$\mathbf{E}^2(\sigma_t)(\mathbf{0}, \underline{a})$ is in good position for each $\underline{a}$. Furthermore the image by $\mathbf{E}^2(\sigma_t)$ of a chain is a chain.*

PROOF. We check it using the graph of Figure 4.4. It is clear that $\mathbf{E}^2(\sigma_t)(\mathbf{0}, \underline{a})$ is in good position for every face which is replaced by only one face, that is, for $(\mathbf{0}, 1\wedge 2), (\mathbf{0}, 1\wedge 3), (\mathbf{0}, 1\wedge 4), (\mathbf{0}, 1\wedge 5), (\mathbf{0}, 3\wedge 4)$. The face $(\mathbf{0}, 2\wedge 4)$ is replaced with a patch of faces $(M_\sigma^{-1}(\mathbf{e}_2 + s\mathbf{e}_1), 3\wedge 5) = (\mathbf{e}_1 + (s-t-1)\mathbf{e}_5, 3\wedge 5)$, for $s = 0, \dots, t$, and $(\mathbf{0}, 1\wedge 3)$. One checks easily that these faces form a chain. Analogously one can check it for $(\mathbf{0}, 3\wedge 5), (\mathbf{0}, 4\wedge 5), (\mathbf{0}, 2\wedge 3)$. By iterating $\mathbf{E}^2(\sigma_t)$ on $(\mathbf{0}, 2\wedge 5)$ we obtain also a perfect matching patch, formed by two chains beginning from $(\mathbf{0}, 1\wedge 4)$ and followed by $(\mathbf{e}_1 - s\mathbf{e}_5, 4\wedge 5)$ and $(\mathbf{e}_4 - s'\mathbf{e}_5, 1\wedge 5)$, for $s = 0, \dots, t$ and $s' = 0, \dots, t-1$ (the latter for $t \geq 1$).

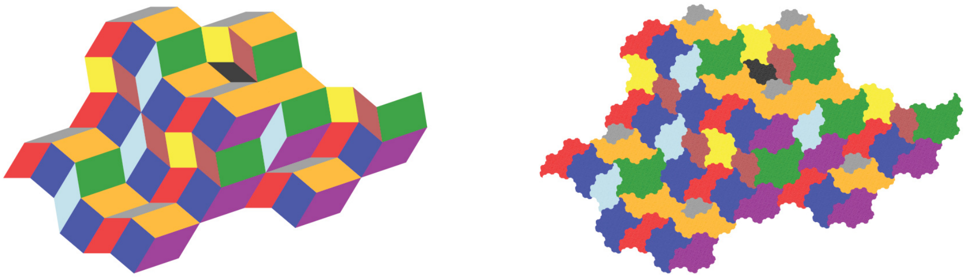


FIGURE 4.5. Polygonal and Rauzy fractals tiling induced by the 5-seed patch $\mathcal{U} = \{(\mathbf{0}, 2 \wedge 3), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 3 \wedge 4)$.

To see that the image by $\mathbf{E}^2(\sigma_t)$ of a chain is a chain it is sufficient to check it for chains made of two faces. Given $(\mathbf{0}, a \wedge b)$ there are only finitely many $(\mathbf{x}, a \wedge c)$ such that together form a chain. With the help of the graph of Figure 4.4 we can make a complete study of all cases and see that the images by $\mathbf{E}^2(\sigma_t)$ of these chains are again chains. □

The patch $\mathcal{U} = \{(\mathbf{0}, 1 \wedge 3), (\mathbf{0}, 1 \wedge 4), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 2 \wedge 5), (\mathbf{0}, 3 \wedge 5)\}$ is an example of a 1-seed patch, that is, $\mathcal{U} \subseteq \mathbf{E}^2(\sigma)(\mathcal{U})$. An example of 5-seed patch is $\mathcal{U} = \{(\mathbf{0}, 2 \wedge 3), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 3 \wedge 4)$.

By Proposition 4.30 we have that σ_t are regular, for all $t \geq 0$, and furthermore the assumption of regularity on the periodic stepped surface is satisfied.

PROPOSITION 4.31. *For every seed patch the geometric finiteness property holds.*

PROOF. We prove that the sufficient condition of Proposition 4.10 for the geometric finiteness property holds. Choose an appropriate orientation such that $M_2' = M_2$. Then, by Lemma 4.17, $\rho(M_2') = \beta_t$, where β_t is the Pisot root of σ_t, $t \in \mathbb{N}_0$. We notice that the matrix M_1' of σ_t has characteristic polynomial $x^5 - tx^4 - (t+1)x - (t+1)$. One can check asymptotically that $\rho(M_1') < t+1 < \rho(M_2')$, for all $t \geq 1$, while for $t = 0$ we have $\rho(M_1') \approx 1,167 < \rho(M_2') \approx 1,325$. □

COROLLARY 4.32. *For every seed patch $\mathcal{U}$ the collection $\mathcal{C}_{\mathcal{U}} = \{\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}) : (\mathbf{x}, \underline{a}) \in \Gamma_{\mathcal{U}}\}$ is an aperiodic self-replicating tiling of $\mathbb{K}_\beta^c \cong \mathbb{C}$. Furthermore, given a 3-patch $\mathcal{P}$, $\mathcal{R}_{\mathcal{P}}$ tiles periodically by the lattice $\pi_c(\Lambda_{\mathcal{P}})$ and the domain exchange $E_{\mathcal{P}}$ is a translation on the torus $\mathbb{C}/\pi_c(\Lambda_{\mathcal{P}})$.*

PROOF. Direct consequence of Theorem 4.20 and Proposition 4.25, since we showed in Proposition 4.30 that the substitutions σ_t are regular and in Proposition 4.31 that the geometric finiteness property holds. □

For example, for σ_0 the 3-touching patch $\mathcal{P} = \{(\mathbf{0}, 2 \wedge 3), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 3 \wedge 4)\}$ induces the periodic tiling $\mathcal{R}_{\mathcal{P}} + \pi_c(\Lambda_{\mathcal{P}})$, with $\Lambda_{\mathcal{P}} = (\mathbf{e}_4 - \mathbf{e}_2)\mathbb{Z} + (\mathbf{e}_4 - \mathbf{e}_3)\mathbb{Z}$.

4.5.1. Decomposition in Hokkaido subtiles. The aim of this section is to find a way to decompose our new Rauzy fractals $\mathcal{R}(a \wedge b)$ in original Dumont-Thomas subtiles $\mathcal{R}(c)$ (see Definition 2.1 and (2.10)).

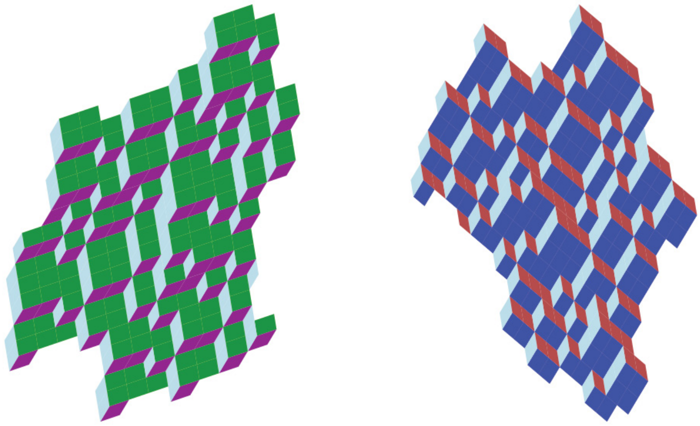


FIGURE 4.6. It is interesting to notice that changing suitably the projection we get different polygonal tilings by some faces of three different types.

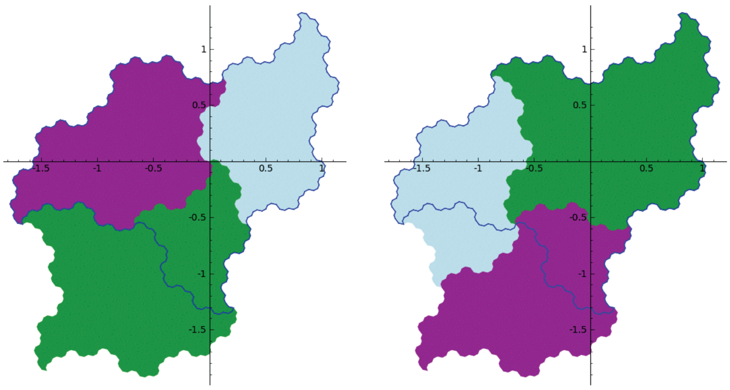


FIGURE 4.7. Domain exchange for $\mathcal{R}(2 \wedge 3)$, $\mathcal{R}(2 \wedge 4)$, $\mathcal{R}(3 \wedge 4)$, and original Hokkaido tile $-\mathcal{R}$.

We can use numeration to describe the subtiles $\mathcal{R}(a \wedge b)$ using the $\mathbf{E}^2(\sigma)$-suffix graph in Figure 4.4. Indeed, by definition

$$\mathcal{R}(a \wedge b) = \Big\{ \sum_{i \geq 0} \delta_c(v_{s_i} \beta^i) : (s_i)_{i \geq 0} \in \mathcal{G}_s(a \wedge b) \Big\}$$

where $\mathcal{G}_s(a \wedge b)$ denotes the set of labels of left-infinite walks in $\mathbf{E}^2(\sigma)$-suffix graph ending at state $a \wedge b$.

Since we are using a suffix description for $\mathcal{R}(a \wedge b)$ it is convenient to express also the Dumont-Thomas subtiles $\mathcal{R}(c)$ in terms of infinite labels of paths in the suffix graph. This can be done as in [CS01b, Section 5] through the use of (1.2), and we get

$$(4.11) \qquad -\mathcal{R}(a) - \delta_c(v_a) = \Big\{ \sum_{i \geq 0} \delta_c(v_{s_i} \beta^i) : (s_i)_{i \geq 0} \in \mathcal{G}_s(a) \Big\},$$

where $\mathcal{G}_s(a)$ denotes the set of labels of left-infinite walks in the suffix graph of the substitution ending at state a (cf. (2.10)). By abuse of notation we will write 0 instead of ϵ by reading labels of walks in the suffix or $\mathbf{E}^2(\sigma_0)$-suffix graphs.

We will relate the elements $\sum_{i\geq 0}\delta_c(v_{s_i}\beta^i)$ with $(s_i)_{i\geq 0}\in\mathcal{G}_s(a\wedge b)$ with those $\sum_{i\geq 0}\delta_c(v_{s_i}\beta^i)$ for $(s_i)_{i\geq 0}\in\mathcal{G}_s(a)$.

We work with the substitution σ_0, generally known with the name of *Hokkaido substitution*, for which we have

$$\mathcal{G}_s(a)=\{(s_i)_{i\geq 0}=\cdots 0^5 2^{k_2} 0^5 2^{k_1} 0^{a-1} : 0\leq k_i\leq\infty\}.$$

Notice that we have $\pi_e(\mathbf{e}_1)=\pi_e(\mathbf{e}_3)+\pi_e(\mathbf{e}_4)$ and $\pi_e(\mathbf{e}_5)=\pi_e(\mathbf{e}_2)+\pi_e(\mathbf{e}_3)$, or equivalently $v_1=v_3+v_4$, $v_5=v_2+v_3$.

LEMMA 4.33. *We have*

$$\begin{aligned}
\mathcal{R}(2\wedge 3)&=(-\mathcal{R}(1)-\delta_c(v_1))\cup(-\mathcal{R}(4)-\delta_c(v_4)),\\
\mathcal{R}(2\wedge 4)&=(-\mathcal{R}(1)-\delta_c(v_3))\cup(-\mathcal{R}(3)-\delta_c(v_3))\cup(-\mathcal{R}(5)-\delta_c(v_3)),\\
\mathcal{R}(3\wedge 4)&=(-\mathcal{R}(2)-\delta_c(v_2))\cup(-\mathcal{R}(5)-\delta_c(v_5)).
\end{aligned}$$

PROOF. Observe that

$$v_2\beta^3=v_2\beta+v_2,\quad\text{i.e.}\quad \lambda(2000.)=\lambda(0022.). \tag{4.12}$$

where $\lambda(w)=\sum_{i=0}^{|w|}\delta_c(v_{w_i}\beta^i)$. Notice that we can extend λ to infinite strings $(s_i)_{i\geq 0}$. We will prove using (4.12) that $\lambda(\mathcal{G}_s(2\wedge 3))=\lambda(\mathcal{G}_s(1)\cup\mathcal{G}_s(4))$. For this reason we will write $w=_\lambda w'$ if $\lambda(w)=\lambda(w')$. The cycle

$$2\wedge 3\xrightarrow{002}2\wedge 5\xrightarrow{00}2\wedge 4\xrightarrow{0}3\wedge 5\overset{\circlearrowleft(20)^k}{}\xrightarrow{00}2\wedge 5\xrightarrow{2}2\wedge 3$$

in the graph of Figure 4.4 produces strings of type $0020^3(20)^k002=_\lambda 0^5 2^{2k+2}$. Starting from state $2\wedge 5$ we get strings $0^5 2^{2k+1}$. Walking from the first node $2\wedge 5$ to the second $2\wedge 5$ returns $0^5 2^{2k}$ and extending this walk to the left starting from $2\wedge 3$ we obtain $0^5 2^{2k+3}$. Walking in Figure 4.4 from $2\wedge 3$ to $2\wedge 5$ we get the word $0022002=_\lambda 20^5 2$. Thus, $\lambda(\mathcal{G}_s(2\wedge 3))=\lambda(\cdots 0^5 2^{k_2} 0^5 2^{k_1})$, with $0\leq k_i\leq\infty$, i.e. $\lambda(\mathcal{G}_s(1))$. Strings ending with 0^3 are obtained following the loop $2\wedge 3\overset{00}{\to}4\wedge 5\overset{2}{\to}2\wedge 5\overset{2}{\to}2\wedge 3$. Since these are all possible paths ending at $2\wedge 3$ we have proven that $\lambda(\mathcal{G}_s(2\wedge 3))=\lambda(\mathcal{G}_s(1)\cup\mathcal{G}_s(4))$ which implies $\mathcal{R}(2\wedge 3)=(-\mathcal{R}(1)-\delta_c(v_1))\cup(-\mathcal{R}(4)-\delta_c(v_4))$ by (4.11).

Since $2\wedge 3$ goes to $3\wedge 4$ by reading a 0 we deduce immediately that all the strings ending at $3\wedge 4$ are equivalent under λ to those in $\mathcal{G}_s(2)\cup\mathcal{G}_s(5)$. Hence by (4.11) we get $\mathcal{R}(3\wedge 4)=(-\mathcal{R}(2)-\delta_c(v_2))\cup(-\mathcal{R}(5)-\delta_c(v_5))$.

Starting from $2\wedge 5$ and going to $2\wedge 4$ passing by $1\wedge 3$ we read 00 and by the above reasonings we get then all possible strings $\cdots 2^{k_2}0^5 2^{k_1}00$ belonging to $\mathcal{G}_s(3)$. From $2\wedge 5\overset{00}{\to}2\wedge 4\overset{0}{\to}3\wedge 5\overset{2}{\to}2\wedge 4$ we get all expansions in $\mathcal{G}_s(5)+2$, where with the latter we mean the set of $(s_i)_{i\geq 0}\in\mathcal{G}_s(5)$ such that $s_0=2$. Walking k times through the loop $2\wedge 4\to 3\wedge 5\to 2\wedge 4$ and extending to the left with $2\wedge 5$ we get strings $0^2(02)^k$. Subtracting $v_4=\lambda(200.)$ we get $0^2(02)^{k-2}0002=_\lambda 0^5 2^{2k-3}$. Walking through the loop $2\wedge 4\to 3\wedge 5\to 2\wedge 5\to 2\wedge 4$ and then once into $2\wedge 4\to 3\wedge 5\to 2\wedge 4$ we read the string $020^5=_\lambda 0^4 2^3$ and, after subtracting v_4, we get $0^5 2^2$. Repeating this loop we get arbitrary large strings ending with an even number of 2s. Thus we have shown we get strings in $\mathcal{G}_s(1) + 4$.

So by (4.11) we just proved that $\mathcal{R}(2 \wedge 4)$ is made of the domains $\lambda(\mathcal{G}_s(3)) = -\mathcal{R}(3) - \delta_c(v_3)$, $\lambda(\mathcal{G}_s(5) + 2) = -\mathcal{R}(5) - \delta_c(v_5) + \delta_c(v_2) = -\mathcal{R}(5) - \delta_c(v_3)$ and $\lambda(\mathcal{G}_s(1) + 4) = -\mathcal{R}(1) - \delta_c(v_1) + \delta_c(v_4) = -\mathcal{R}(1) - \delta_c(v_3)$, since $v_1 = v_3 + v_4$ and $v_5 = v_2 + v_3$. □

We can carry on the computations for the other $\mathcal{R}(a \wedge b)$ in a similar way as above. We obtain

$$\mathcal{R}(2 \wedge 5) = (-\mathcal{R}(1) - \delta_c(v_1)) \cup (-\mathcal{R}(4) - \delta_c(v_4) + \delta_c(v_2)),$$
$$\mathcal{R}(3 \wedge 5) = (-\mathcal{R}(5) - \delta_c(v_5)) \cup (-\mathcal{R}(1) - \delta_c(v_4)),$$

and for the others just use the set equation $\mathcal{R}(a \wedge b) = \beta \mathcal{R}(a' \wedge b')$ for $a \wedge b \leftarrow a' \wedge b'$.

Similar formulas to express the Rauzy fractals in terms of the subtiles $\mathcal{R}(a)$ hold for the entire family of substitutions σ_t.

4.5.2. Coding of the domain exchange and broken lines. We describe an interesting interpretation for $\mathcal{R}_{\mathcal{P}}$ in terms of broken lines. Being reducible for a substitution means that we have some linear dependencies between the $\pi(\mathbf{e}_i)$, for $i = 1, \dots, n$. For Hokkaido we have

$$\pi(\mathbf{e}_1) = \pi(\mathbf{e}_3) + \pi(\mathbf{e}_4), \qquad \pi(\mathbf{e}_5) = \pi(\mathbf{e}_2) + \pi(\mathbf{e}_3). \tag{4.13}$$

We have a broken line in $\mathbb{R}^5$ which is the geometrical interpretation of the fixed point u of σ:

$$\overline{u} = \bigcup_{i \geq 1} \{(\mathbf{l}(u_{[0,i)}), u_i)\},$$

where $u_{[0,i)} = u_0 \cdots u_{i-1}$ and $(\mathbf{x}, i)$ denotes the segment from $\mathbf{x}$ to $\mathbf{x} + \mathbf{e}_i$. Projecting the broken line into $\mathbb{K}_\beta \cong \mathbb{R}^3$ the rational dependencies show up, and we get what we call a "reducible" broken line, made of five different segments. We can change this broken line using the rational dependencies, i.e. we substitute every $\pi(\mathbf{e}_1)$ and $\pi(\mathbf{e}_5)$ with their linearly independent atoms $\pi(\mathbf{e}_2)$, $\pi(\mathbf{e}_3)$ and $\pi(\mathbf{e}_4)$ as in the relations (4.13). Combinatorially this is equivalent to applying the code

$$\chi: \ 1 \mapsto 34, \quad 2 \mapsto 2, \quad 3 \mapsto 3, \quad 4 \mapsto 4, \quad 5 \mapsto 32, \tag{4.14}$$

to the fixed point of σ. Operating in this way we get more vertices in the new broken line, included the old ones. View as a tiling of the line, we get a decomposition of the tiling of the line using only the rationally independent lengths v_2, v_3 and v_4.

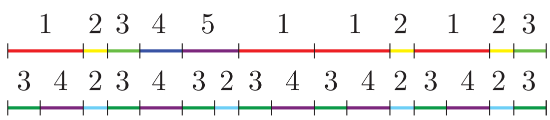


FIGURE 4.8. Effect of the code χ on the tiling of the line determined by the fixed point of σ_0.

So we may say that in this process we converted the substitution in an irreducible one. A similar approach was considered for a Pisot substitutions on four letters in [Fre05, Section 2.2, 2.3] in the framework of model sets.

Let $w = \chi(u)$ and consider the *refinements*

$$\widetilde{\mathcal{R}}(a_{[0,\ell)}) := \overline{\{\pi_c(\mathbf{l}(w_{[0,N)}) : N \in \mathbb{N}, w_{[N,N+\ell)} = a_{[0,\ell)}\}}, \tag{4.15}$$

for $a_{[0,\ell)} \in \{2,3,4\}^*$. Using the definition (2.11) of the subtiles $\mathcal{R}(a)$ as projections of coloured vertices of the broken line and considering the refinements $\widetilde{\mathcal{R}}(a) = \widetilde{\mathcal{R}}(a_{[0,1)})$, we see from the code χ that

$$\mathcal{R}(1) \subset \widetilde{\mathcal{R}}(3),\ \mathcal{R}(1) + \delta_c(v_3) \subset \widetilde{\mathcal{R}}(4), \quad \mathcal{R}(5) \subset \widetilde{\mathcal{R}}(3),\ \mathcal{R}(5) + \delta_c(v_3) \subset \widetilde{\mathcal{R}}(2)$$

and of course $\mathcal{R}(2) \subset \widetilde{\mathcal{R}}(2)$, $\mathcal{R}(3) \subset \widetilde{\mathcal{R}}(3)$, $\mathcal{R}(4) \subset \widetilde{\mathcal{R}}(4)$. Therefore, in view of Lemma 4.33, we can deduce the following.

Proposition 4.34. *We have*

$$\widetilde{\mathcal{R}}(a) + \delta_c(v_a) = -\mathcal{R}(b \wedge c), \quad \textit{for} \quad \{b,c\} = \{2,3,4\} \setminus \{a\}.$$

Let $\widetilde{\mathcal{R}} = \bigcup_{a\in\{2,3,4\}} \widetilde{\mathcal{R}}(a)$. Denote by $\widetilde{E}$ the domain exchange $\widetilde{\mathcal{R}} \to \widetilde{\mathcal{R}}$, $\mathbf{x} \mapsto \mathbf{x} + \delta_c(v_a)$, for $\mathbf{x} \in \widetilde{\mathcal{R}}(a)$. Then $(\widetilde{\mathcal{R}}, \widetilde{E}) = (-\mathcal{R}_{\mathcal{P}}, E_{\mathcal{P}}^{-1})$.

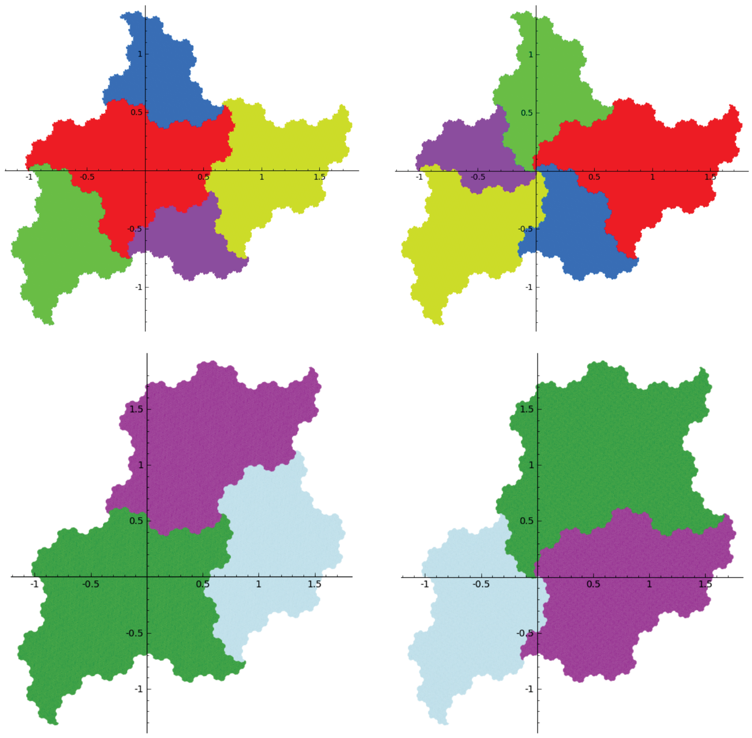


Figure 4.9. The domain exchanges E on the original Hokkaido tile $\mathcal{R}$ and $\widetilde{E}$ on $\widetilde{\mathcal{R}}$.

This is in fact not surprising since we considered suffixes in the definition of $\mathbf{E}^{d-1}(\sigma)$ and we constructed the $\widetilde{\mathcal{R}}(a)$ as projection of vertices of the broken line, i.e. considering prefixes. In fact these two constructions are equivalent up to isometry, as it is explained also in [CS01b, Section 5].

Combinatorially the code χ describes the first return of $\widetilde{E}$ on $\mathcal{R}$. Indeed from the decomposition of Lemma 4.33 and Proposition 4.34 one can see in Figure 4.9

for example that $\mathcal{R}(1) \subset \widetilde{\mathcal{R}}(3)$ before returning to $\mathcal{R}$ under $\widetilde{E}$ passes through $\widetilde{\mathcal{R}}(4)$.

Proposition 4.35. *E is the first return of $\widetilde{E}$ on $\mathcal{R}$.*

Proof. From the decomposition of Lemma 4.33 and Proposition 4.34 we see that $\widetilde{E}(\widetilde{\mathcal{R}}(4)) = \widetilde{\mathcal{R}}(4) + \delta_c(v_4) = (\mathcal{R}(1) + \delta_c(v_3 + v_4)) \cup (\mathcal{R}(4) + \delta_c(v_4))$ and $\widetilde{E}(\widetilde{\mathcal{R}}(2)) = \widetilde{\mathcal{R}}(2) + \delta_c(v_2) = (\mathcal{R}(2) + \delta_c(v_2)) \cup (\mathcal{R}(5) + \delta_c(v_3 + v_2))$. Furthermore $\mathcal{R}(1)$, $\mathcal{R}(3)$ and $\mathcal{R}(5)$ are in $\widetilde{\mathcal{R}}(3)$ and applying $\widetilde{E}$ we translate them by $\delta_c(v_3)$. Using the relations $v_3 + v_4 = v_1$ and $v_2 + v_3 = v_5$ we get $\widetilde{E}|_{\mathcal{R}} = E$. □

Let $\Omega = \overline{\{S^k w : k \in \mathbb{N}\}}$, where $w = \chi(u)$ is the coded fixed point of σ.

Lemma 4.36. *(Ω, S) is minimal and uniquely ergodic.*

Proof. We know that (X_σ, S), $X_\sigma = \overline{\{S^k u : k \in \mathbb{N}\}}$, is minimal and uniquely ergodic. In particular every factor of u occurs in u with bounded gaps, and the same happens for $\chi(u)$. Thus (X_σ, S) is minimal. Applying the code χ to the word u changes linearly the frequencies of the letters $2, 3, 4$, and we can associate uniquely an invariant measure m, such that its value on the cylinder $[a] := \{w' \in \Omega : w'_0 = a\}$, for $a \in \{2, 3, 4\}$, is exactly the frequency of the letter a (for more details see [Que10]). Hence (Ω, S) is uniquely ergodic. □

We want to show that the dynamical system (Ω, S, m), where m is the unique S-invariant Borel probability measure on Ω, is measurably conjugate to $(\widetilde{\mathcal{R}}, \widetilde{E}, \mu_c)$. Following [BST14], we define the *representation map*

$$\phi : \Omega \to \widetilde{\mathcal{R}}, \quad (a_i)_{i\in\mathbb{N}} \mapsto \bigcap_{\ell\in\mathbb{N}} \widetilde{\mathcal{R}}(a_{[0,\ell)}). \tag{4.16}$$

Lemma 4.37. *ϕ is well-defined, continuous and surjective.*

Proof. Let $(a_i)_{i\in\mathbb{N}} \in \Omega$. Then $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}(a_{[0,0)}) \supset \widetilde{\mathcal{R}}(a_{[0,1)}) \supset \cdots$, and $\widetilde{\mathcal{R}}(a_{[0,\ell)}) \neq \emptyset$ for all $\ell \in \mathbb{N}$. The word $w = \chi(u)$ is uniformly recurrent, since u is generated by a primitive substitution and χ does not affect the uniformly recurrence. Thus we have a sequence $(\ell_k)_{k\in\mathbb{N}}$ such that $a_{[\ell_k,\ell_k+k)} = w_{[0,k)}$, for all $k \in \mathbb{N}$. Since $\widetilde{\mathcal{R}}(a_{[0,\ell_k+k)}) \subset \widetilde{\mathcal{R}}(a_{[\ell_k,\ell_k+k)}) - \pi_c(\mathbf{l}(a_{[0,\ell_k)}))$, we need to show that the diameter of $\widetilde{\mathcal{R}}(a_{[\ell_k,\ell_k+k)}) = \widetilde{\mathcal{R}}(w_{[0,k)})$ converges to zero. Let $\mathcal{S}_k = \{\pi_c(\mathbf{l}(w_{[0,j)}) : 0 \leq j \leq k\}$. Then $\widetilde{\mathcal{R}}(w_{[0,k)}) + \mathcal{S}_k \subset \widetilde{\mathcal{R}}$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} \mathcal{S}_k = \widetilde{\mathcal{R}}$ with respect to the Hausdorff metric. But this implies that $\lim_{k\to\infty} \widetilde{\mathcal{R}}(w_{[0,k)}) = \{\mathbf{0}\}$, which proves that ϕ is well-defined.

The map ϕ is continuous since the sequence $(\widetilde{\mathcal{R}}(a_{[0,\ell)}))_{\ell\in\mathbb{N}}$ is nested and converges to a single point. The surjectivity follows from a Cantor diagonal argument. □

Lemma 4.38. *(Ω, S, m) is measurably conjugate to $(\widetilde{\mathcal{R}}, \widetilde{E}, \mu_c)$.*

Proof. The collections $\mathcal{K}_i = \{\widetilde{\mathcal{R}}(a_{[0,i)}) : a_{[0,i)} \in L_i(w)\}$ are measure-theoretic partitions of $\widetilde{\mathcal{R}}$ since the strong coincidence condition for $\mathcal{P}$ holds and $\widetilde{\mathcal{R}}(a_{[0,i)}) = \bigcap_{j=0}^{i-1} \widetilde{E}^{-j} \widetilde{\mathcal{R}}(a_j)$. Hence $\phi(w') \neq \phi(w'')$ for all $w', w'' \in \phi^{-1}(\widetilde{\mathcal{R}} \setminus \bigcup_{i\in\mathbb{N}, K\in\mathcal{K}_i} \partial K)$ with $w' \neq w''$, and $m(\phi^{-1}(\bigcup_{i\in\mathbb{N}, K\in\mathcal{K}_i} \partial K)) = 0$ since we have $\mu_c(\partial K) = 0$ for all

$K \in \mathcal{K}_i$, $i \in \mathbb{N}$, by Proposition 4.19, and $\mu_c \circ \phi$ is an S-invariant Borel measure, which equals m by unique ergodicity of (Ω, S). Thus the map ϕ is injective almost everywhere. Finally $\phi((a_k)_{k\in\mathbb{N}})$ is a single point $\mathbf{z} = \bigcap_{\ell\in\mathbb{N}} \widetilde{\mathcal{R}}(a_{[0,\ell)})$. Since $\widetilde{\mathcal{R}}(a_{[0,\ell+1)}) + \pi_c(\mathbf{e}_{a_0}) \subset \widetilde{\mathcal{R}}(a_{[1,\ell+1)})$, for all $\ell \in \mathbb{N}$, we obtain that $\widetilde{E}(\mathbf{z}) = \mathbf{z} + \pi_c(\mathbf{e}_{a_0}) = \bigcap_{\ell\in\mathbb{N}} \widetilde{\mathcal{R}}(a_{[1,\ell)})$, but this is the same as shifting $(a_k)_{k\in\mathbb{N}}$ and applying ϕ. Thus we checked that $\widetilde{E} \circ \phi = \phi \circ S$. □

Notice that the relations (4.13) hold for the whole family of σ_t. The interpretation with broken lines (4.15) and the results of Proposition 4.34, 4.35 and Lemma 4.38 can be generalized for the entire family of substitutions σ_t, $t \in \mathbb{N}_0$.

THEOREM 4.39. *For the family of substitutions σ_t, $t \in \mathbb{N}_0$, we have the following commutative diagram:*

$$\begin{array}{ccccccc}
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \widetilde{\mathcal{R}} & \longrightarrow & \mathbb{C}/\Lambda \\
\downarrow S & & \downarrow S & & \downarrow \widetilde{E} & & \downarrow \widetilde{E} \\
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \widetilde{\mathcal{R}} & \longrightarrow & \mathbb{C}/\Lambda
\end{array}$$

where χ is the code (4.14).

4.6. Non-regular examples

Consider the substitution

$$\sigma: \ 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 4, \ 4 \mapsto 1,$$

with characteristic polynomial $f(x)g(x) = (x^3 - 2x^2 + x - 1)(x+1)$. The rational dependency relation is $v_1 + v_3 = v_2 + v_4$. Since $n = 4$ and $d = 3$, we deal with $\mathbf{E}_2^*(\sigma)$ and its geometric realization $\mathbf{E}^2(\sigma)$.

PROPOSITION 4.40. *The substitution σ is non-regular.*

PROOF. We observe that $\mathbf{E}^2(\sigma)(\mathbf{0}, 2\wedge 4) = (\mathbf{0}, 1\wedge 3) + (\mathbf{e}_1 - \mathbf{e}_4, 3\wedge 4)$ is not in good position since the projections of the two faces do not have disjoint interiors (see Figure 4.10). □

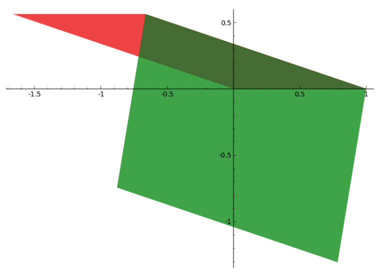


FIGURE 4.10. The overlapping faces $(\delta_c(0), 1\wedge 3)$ and $(\delta_c(v_1 - v_4), 3\wedge 4)$.

Overlaps can be observed for the whole family of substitutions

$$\sigma_t: 1 \mapsto 1^{t-1}2, \ 2 \mapsto 1^{t-1}3, \ 3 \mapsto 4, \ 4 \mapsto 1 \quad (t \geq 2).$$

See also [Fur06] for some similar polygonal overlaps obtained in the framework of non-Pisot unimodular matrices.

The overlaps problem could be solved by considering a different projection onto $\mathbb{K}_\beta^c$, or by operating some flips on the problematic faces of the stepped surface (for a treatment of flips on stepped surfaces see e.g. [ABFJ07, BF11]). Nevertheless, we notice that in our case the overlaps would vanish in the Hausdorff limit process generating the Rauzy fractals, since the resulting overlapping polygons have only two small shapes and occur distant to each other.

We notice that for this family of substitutions the Rauzy fractal $\mathcal{R}$, defined in Chapter 2, provides a periodic tiling related to the domain exchange transformation, since the quotient mapping condition (QM) holds. For the family of Section 4.5 this condition does not hold but nevertheless we obtain new periodic tilings using some $\mathcal{R}(a \wedge b)$.

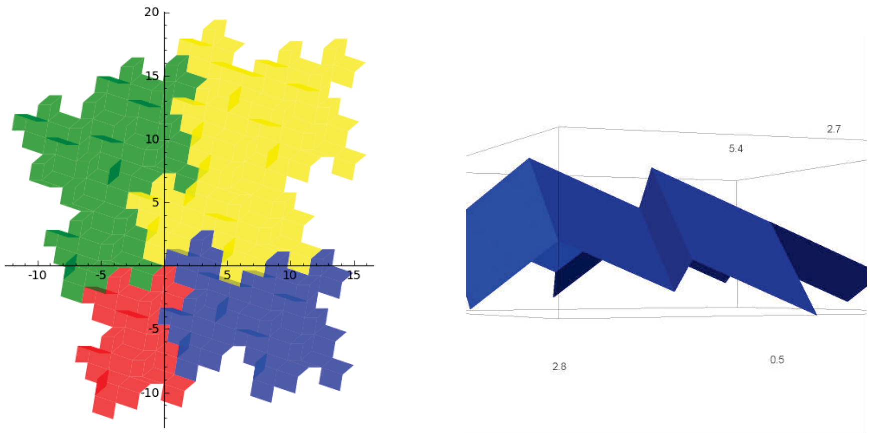


FIGURE 4.11. Projected $\mathbf{E}^2(\sigma)$-iterate of a patch and explanation of the overlaps: this is due to the particular "inclination" of two types of faces in $\mathbb{K}_\beta$, in the sense that they insert themselves under other faces, producing overlaps when projecting them along the expanding direction.

Conclusion

We have seen that enlarging the representation space with $\mathfrak{p}$-adic factors we can set up a Rauzy fractal theory even in the non-unit case and we have substantially the same generalized ingredients as in the unit case to attack Pisot conjecture. Leaving irreducibility involves new combinatorial constructions to see the substitution as if it were irreducible, as well as higher dimensional duals of geometric realization of substitutions. Especially the last constructions and ideas lead to many perspectives.

- Some initial questions are: can we generalize the constructions of Chapter 4 to any reducible Pisot substitution? Is it possible to characterize the points of our stepped surfaces in a similar way as in the irreducible case? Can we improve some tiling results discarding or improving some of the hypotheses we needed? Another important point is to investigate the dynamics and the geometry in the neutral space of the substitution, and understand whether they have some influence in the Rauzy fractals tiling theory. Some studies in this direction have been initiated in [ABB11]. The study of the combinatorial codes and of the new modified broken lines deserves more investigations. Figure 4.6 suggests that we can change projection and view stepped surfaces as if we were in the irreducible settings, by selecting some of the faces, getting rid of the linear dependencies. Can we in general reduce the study of the dynamics of a reducible substitution to that of an irreducible one? We saw that for a family of substitutions (X_σ, S) can be interpreted as the first return of a toral translation. Induced dynamics can have different behaviour, thus it is natural to ask in which cases these first returns have pure discrete spectrum (see [Rau84]). Finally further research could be motivated by the fact that the existence of polygonal approximations for the Rauzy fractals allows now to define contact graphs even in the reducible case.

- Can we isolate some classes of reducible substitutions whose associated substitutive systems have pure discrete spectrum? And when is it true in general that, like constant length substitutions, reducible substitutive systems are codings of skew products of rotations on tori or solenoids and finite groups?

- In view of the coincidence rank conjecture, can we define a suitable cohomology on the Rauzy fractals tiling spaces? Since it is conjectured that the coincidence rank divides the norm of the Pisot number, are there any examples of non-unit irreducible Pisot substitutions for which Pisot conjecture does not hold?

- In this thesis we dealt with symbolic dynamical systems generated by a single substitution. A fractal theory associated with $\boldsymbol{S}$-adic subshifts, where $\boldsymbol{S}$ is a finite set of unit Pisot substitutions, has been developed in [BST14]. To which extent can we generalize this theory to the non-unit reducible case?

- Sturmian words are the codings of rotations on one-dimensional tori and their connection with continued fractions is well-known. It could be possible to generalize to higher dimensions and investigate the connections between substitution dynamical systems, Rauzy fractals and multi-dimensional continued fractions, with the help of the Teichmüller geodesic flow, as done in the one-dimensional case in [AF01].

Bibliography

[ABB11] P. Arnoux, J. Bernat, and X. Bressaud, *Geometrical models for substitutions*, Exp. Math. **20** (2011), no. 1, 97–127. (page 97)

[ABB+14] S. Akiyama, M. Barge, V. Berthé, J.-Y. Lee, and A. Siegel, *On the Pisot substitution conjecture*, preprint, 2014. (page 8)

[ABBS08] S. Akiyama, G. Barat, V. Berthé, and A. Siegel, *Boundary of central tiles associated with Pisot beta-numeration and purely periodic expansions*, Monatsh. Math. **155** (2008), no. 3-4, 377–419. (page 9, 12, 14, 33, 42, 48, 54, 55, 56, 71, 72)

[ABFJ07] P. Arnoux, V. Berthé, T. Fernique, and D. Jamet, *Functional stepped surfaces, flips, and generalized substitutions*, Theoret. Comput. Sci. **380** (2007), no. 3, 251–265. (page 85, 96)

[Adl98] R. L. Adler, *Symbolic dynamics and Markov partitions*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 1, 1–56. (page 5)

[AF01] P. Arnoux and A. M. Fisher, *The scenery flow for geometric structures on the torus: the linear setting*, Chinese Ann. Math. Ser. B **22** (2001), no. 4, 427–470. (page 98)

[AFHI11] P. Arnoux, M. Furukado, E. Harriss, and S. Ito, *Algebraic numbers, free group automorphisms and substitutions on the plane*, Trans. Amer. Math. Soc. **363** (2011), no. 9, 4651–4699. (page 6, 14, 78, 79)

[AFSS10] B. Adamczewski, C. Frougny, A. Siegel, and W. Steiner, *Rational numbers with purely periodic β-expansion*, Bull. Lond. Math. Soc. **42** (2010), no. 3, 538–552. (page 14)

[AI01] P. Arnoux and S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. Simon Stevin **8** (2001), no. 2, 181–207, Journées Montoises d'Informatique Théorique (Marne-la-Vallée, 2000). (page 4, 12, 18, 38)

[Aki98] S. Akiyama, *Pisot numbers and greedy algorithm*, Number theory (Eger, 1996), de Gruyter, Berlin, 1998, pp. 9–21. (page 14, 55, 72)

[Aki00] ———, *Cubic Pisot units with finite beta expansions*, Algebraic number theory and Diophantine analysis (Graz, 1998), de Gruyter, Berlin, 2000, pp. 11–26. (page 46)

[Aki02] ———, *On the boundary of self affine tilings generated by Pisot numbers*, J. Math. Soc. Japan **54** (2002), no. 2, 283–308. (page 7, 33, 47, 66)

[AL11] S. Akiyama and J.-Y. Lee, *Algorithm for determining pure pointedness of self-affine tilings*, Adv. Math. **226** (2011), no. 4, 2855–2883. (page 7)

[ARS04] S. Akiyama, H. Rao, and W. Steiner, *A certain finiteness property of Pisot number systems*, J. Number Theory **107** (2004), no. 1, 135–160. (page 7, 56, 66)

[AW70] R. L. Adler and B. Weiss, *Similarity of automorphisms of the torus*, Memoirs of the American Mathematical Society, No. 98, American Mathematical Society, Providence, R.I., 1970. (page 5)

[Bar88] M. Barnsley, *Fractals everywhere*, Academic Press Inc., Boston, MA, 1988. (page 3)

[Bar13] M. Barge, *Factors of Pisot tiling spaces and the coincidence rank conjecture*, preprint, arXiv:1301.7094, 2013. (page 11)

[Bar14] ———, *Pure discrete spectrum for a class of one-dimensional substitution tiling systems*, preprint, arXiv:1403.7826, 2014. (page 7)

[BBJS12] M. Barge, H. Bruin, L. Jones, and L. Sadun, *Homological Pisot substitutions and exact regularity*, Israel J. Math. **188** (2012), 281–300. (page 11, 83)

[BBJS14] V. Berthé, J. Bourdon, T. Jolivet, and A. Siegel, *A combinatorial approach to products of Pisot substitutions*, preprint, arXiv:1401.0704, 2014. (page 8)

[BBK06] V. Baker, M. Barge, and J. Kwapisz, *Geometric realization and coincidence for reducible non-unimodular Pisot tiling spaces with an application to β-shifts*, Ann. Inst.

Fourier (Grenoble) **56** (2006), no. 7, 2213–2248, Numération, pavages, substitutions. (page 10, 11, 15, 25, 66)

[BBLT06] G. Barat, V. Berthé, P. Liardet, and J. Thuswaldner, *Dynamical directions in numeration*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 7, 1987–2092, Numération, pavages, substitutions. (page 21)

[BD02] M. Barge and B. Diamond, *Coincidence for substitutions of Pisot type*, Bull. Soc. Math. France **130** (2002), no. 4, 619–626. (page 4, 18)

[Ber77] A. Bertrand, *Développements en base de Pisot et répartition modulo* 1, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 6, A419–A421. (page 24, 54)

[BF11] V. Berthé and T. Fernique, *Brun expansions of stepped surfaces*, Discrete Math. **311** (2011), no. 7, 521–543. (page 85, 96)

[BG13] M. Baake and U. Grimm, *Aperiodic order. Vol. 1*, Encyclopedia of Mathematics and its Applications, vol. 149, Cambridge University Press, Cambridge, 2013, A mathematical invitation, With a foreword by Roger Penrose. (page 5, 39)

[BK05] M. Barge and J. Kwapisz, *Elements of the theory of unimodular Pisot substitutions with an application to β-shifts*, Algebraic and topological dynamics, Contemp. Math., vol. 385, Amer. Math. Soc., Providence, RI, 2005, pp. 89–99. (page 66)

[BK06] ———, *Geometric theory of unimodular Pisot substitutions*, Amer. J. Math. **128** (2006), no. 5, 1219–1282. (page 7)

[Bla89] F. Blanchard, *β-expansions and symbolic dynamics*, Theoret. Comput. Sci. **65** (1989), no. 2, 131–141. (page 24)

[BM86] A. Bertrand-Mathis, *Développement en base θ; répartition modulo un de la suite $(x\theta^n)_{n\geq 0}$; langages codés et θ-shift*, Bull. Soc. Math. France **114** (1986), no. 3, 271–323. (page 24)

[BM04] M. Baake and R. V. Moody, *Weighted Dirac combs with pure point diffraction*, J. Reine Angew. Math. **573** (2004), 61–94. (page 39)

[Bow70] R. Bowen, *Markov partitions for Axiom* A *diffeomorphisms*, Amer. J. Math. **92** (1970), 725–747. (page 5)

[Bow78] ———, *Markov partitions are not smooth*, Proc. Amer. Math. Soc. **71** (1978), no. 1, 130–132. (page 5)

[BS02] M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002. (page 5)

[BS05] V. Berthé and A. Siegel, *Tilings associated with beta-numeration and substitutions*, Integers **5** (2005), no. 3, A2, 46. (page 8, 36)

[BS07] ———, *Purely periodic β-expansions in the Pisot non-unit case*, J. Number Theory **127** (2007), no. 2, 153–172. (page 6, 9, 13, 54, 55)

[BSS+11] V. Berthé, A. Siegel, W. Steiner, P. Surer, and J. M. Thuswaldner, *Fractal tiles associated with shift radix systems*, Adv. Math. **226** (2011), no. 1, 139–175. (page 13, 53, 55)

[BST10] V. Berthé, A. Siegel, and J. Thuswaldner, *Substitutions, Rauzy fractals and tilings*, Combinatorics, automata and number theory, Encyclopedia Math. Appl., vol. 135, Cambridge Univ. Press, Cambridge, 2010, pp. 248–323. (page 4, 8, 45, 46)

[BST14] V. Berthé, W. Steiner, and J.M. Thuswaldner, *Geometry, dynamics and arithmetic of $\boldsymbol{S}$-adic shifts*, in preparation, 2014. (page 94, 98)

[BT87] E. Bombieri and J. E. Taylor, *Quasicrystals, tilings, and algebraic number theory: some preliminary connections*, The legacy of Sonya Kovalevskaya (Cambridge, Mass., and Amherst, Mass., 1985), Contemp. Math., vol. 64, Amer. Math. Soc., Providence, RI, 1987, pp. 241–264. (page 2)

[Cas67] J. W. S. Cassels, *Global fields*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 42–84. (page 26, 30)

[CEW97] V. Chothi, G. Everest, and T. Ward, *S-integer dynamical systems: periodic points*, J. Reine Angew. Math. **489** (1997), 99–132. (page 9)

[CFS82] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 245, Springer-Verlag, New York, 1982. (page 2, 18)

[Coh93] H. Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics, vol. 138, Springer-Verlag, Berlin, 1993. (page 30)

[CS01a] V. Canterini and A. Siegel, *Automate des préfixes-suffixes associé à une substitution primitive*, J. Théor. Nombres Bordeaux **13** (2001), no. 2, 353–369. (page 19, 44)

[CS01b] ———, *Geometric representation of substitutions of Pisot type*, Trans. Amer. Math. Soc. **353** (2001), no. 12, 5121–5144. (page 4, 18, 44, 90, 93)

[CS03] A. Clark and L. Sadun, *When size matters: subshifts and their related tiling spaces*, Ergodic Theory Dynam. Systems **23** (2003), no. 4, 1043–1057. (page 2)

[Dek78] F. M. Dekking, *The spectrum of dynamical systems arising from substitutions of constant length*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **41** (1977/78), no. 3, 221–239. (page 1)

[DK78] F. M. Dekking and M. Keane, *Mixing properties of substitutions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **42** (1978), no. 1, 23–33. (page 2)

[DK02] K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs, vol. 29, Mathematical Association of America, Washington, DC, 2002. (page 24)

[DT89] J.-M. Dumont and A. Thomas, *Systemes de numeration et fonctions fractales relatifs aux substitutions*, Theoret. Comput. Sci. **65** (1989), no. 2, 153–169. (page 20, 21)

[Dur10] F. Durand, *Combinatorics on Bratteli diagrams and dynamical systems*, Combinatorics, automata and number theory, Encyclopedia Math. Appl., vol. 135, Cambridge Univ. Press, Cambridge, 2010, pp. 324–372. (page 45)

[EI05] H. Ei and S. Ito, *Tilings from some non-irreducible, Pisot substitutions*, Discrete Math. Theor. Comput. Sci. **7** (2005), no. 1, 81–121 (electronic). (page 10, 15, 69)

[EIR06] H. Ei, S. Ito, and H. Rao, *Atomic surfaces, tilings and coincidences. II. Reducible case*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 7, 2285–2313, Numération, pavages, substitutions. (page 10, 69, 81, 83)

[ES97] M. Einsiedler and K. Schmidt, *Markov partitions and homoclinic points of algebraic* $\mathbf{Z}^d$*-actions*, Tr. Mat. Inst. Steklova **216** (1997), no. Din. Sist. i Smezhnye Vopr., 265–284. (page 6)

[EW11] M. Einsiedler and T. Ward, *Ergodic theory with a view towards number theory*, Graduate Texts in Mathematics, vol. 259, Springer-Verlag London Ltd., London, 2011. (page 2, 18)

[Fab95] S. Fabre, *Substitutions et* β*-systèmes de numération*, Theoret. Comput. Sci. **137** (1995), no. 2, 219–236. (page 69)

[Fal03] K. Falconer, *Fractal geometry*, second ed., John Wiley & Sons Inc., Hoboken, NJ, 2003, Mathematical foundations and applications. (page 3)

[Fer06] T. Fernique, *Multidimensional Sturmian sequences and generalized substitutions*, Internat. J. Found. Comput. Sci. **17** (2006), no. 3, 575–599. (page 85)

[FGK03] C. Frougny, J.-P. Gazeau, and R. Krejcar, *Additive and multiplicative properties of point sets based on beta-integers*, Theoret. Comput. Sci. **303** (2003), no. 2-3, 491–516. (page 69)

[FMN96] S. Ferenczi, C. Mauduit, and A. Nogueira, *Substitution dynamical systems: algebraic characterization of eigenvalues*, Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 4, 519–533. (page 2)

[Fog02] N. P. Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002, Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel. (page 2, 11, 18)

[Fre05] D. Frettlöh, *Duality of model sets generated by substitutions*, Rev. Roumaine Math. Pures Appl. **50** (2005), no. 5-6, 619–639. (page 92)

[FS92] C. Frougny and B. Solomyak, *Finite beta-expansions*, Ergodic Theory Dynam. Systems **12** (1992), no. 4, 713–723. (page 7, 13, 46)

[Fur06] M. Furukado, *Tilings from non-Pisot unimodular matrices*, Hiroshima Math. J. **36** (2006), no. 2, 289–329. (page 96)

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. (page 78)

[HI97] M. Hama and T. Imahashi, *Periodic* β*-expansions for certain classes of Pisot numbers*, Comment. Math. Univ. St. Paul. **46** (1997), no. 2, 103–116. (page 6, 13, 54)

[Hol96] M. Hollander, *Linear numeration systems, finite β-expansions, and discrete spectrum for substitution dynamical systems*, Ph.D. thesis, University of Washington, 1996. (page 66)

[Hos86] B. Host, *Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable*, Ergodic Theory Dynam. Systems **6** (1986), no. 4, 529–540. (page 2)

[HS03] M. Hollander and B. Solomyak, *Two-symbol Pisot substitutions have pure discrete spectrum*, Ergodic Theory Dynam. Systems **23** (2003), no. 2, 533–540. (page 6)

[IO93] S. Ito and M. Ohtsuki, *Modified Jacobi-Perron algorithm and generating Markov partitions for special hyperbolic toral automorphisms*, Tokyo J. Math. **16** (1993), no. 2, 441–472. (page 4)

[IO94] ———, *Parallelogram tilings and Jacobi-Perron algorithm*, Tokyo J. Math. **17** (1994), no. 1, 33–58. (page 4)

[IR05] S. Ito and H. Rao, *Purely periodic β-expansions with Pisot unit base*, Proc. Amer. Math. Soc. **133** (2005), no. 4, 953–964. (page 6, 13, 54)

[IR06] S. Ito and H. Rao, *Atomic surfaces, tilings and coincidence. I. Irreducible case*, Israel J. Math. **153** (2006), 129–155. (page 4, 6, 7, 13, 14, 64, 69, 70)

[IT74] S. Ito and Y. Takahashi, *Markov subshifts and realization of β-expansions*, J. Math. Soc. Japan **26** (1974), 33–55. (page 24)

[JLL13] T. Jolivet, B. Loridant, and J. Luo, *Rauzy fractals with countable fundamental group*, preprint, `arXiv:1312.7829`, 2013. (page 4)

[Ken92] R. Kenyon, *Self-replicating tilings*, Symbolic dynamics and its applications (New Haven, CT, 1991), Contemp. Math., vol. 135, Amer. Math. Soc., Providence, RI, 1992, pp. 239–263. (page 5)

[KH95] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza. (page 5)

[Knu98] D. E. Knuth, *The art of computer programming. Vol. 2*, Addison-Wesley, Reading, MA, 1998, Seminumerical algorithms, Third edition [of MR0286318]. (page 19)

[KS12] C. Kalle and W. Steiner, *Beta-expansions, natural extensions and multiple tilings associated with Pisot units*, Trans. Amer. Math. Soc. **364** (2012), no. 5, 2281–2318. (page 28, 54, 55, 59, 60, 64, 66)

[Kur66] K. Kuratowski, *Topology. Vol. I*, New edition, revised and augmented. Translated from the French by J. Jaworowski, Academic Press, New York, 1966. (page 21)

[KV98] R. Kenyon and A. Vershik, *Arithmetic construction of sofic partitions of hyperbolic toral automorphisms*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 357–372. (page 6)

[LB95] S. Le Borgne, *Un codage sofique des automorphismes hyperboliques du tore*, Séminaires de Probabilités de Rennes (1995), Publ. Inst. Rech. Math. Rennes, vol. 1995, Univ. Rennes I, Rennes, 1995, p. 35. (page 6)

[Lin84] D. A. Lind, *The entropies of topological Markov shifts and a related class of algebraic integers*, Ergodic Theory Dynam. Systems **4** (1984), no. 2, 283–300. (page 2)

[Liv87] A. N. Livshits, *On the spectra of adic transformations of Markov compact sets*, Uspekhi Mat. Nauk **42** (1987), no. 3(255), 189–190. (page 7)

[Liv92] ———, *Some examples of adic transformations and automorphisms of substitutions*, Selecta Math. Soviet. **11** (1992), no. 1, 83–104, Selected translations. (page 7)

[LM95] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995. (page 5, 18)

[LMS02] J.-Y. Lee, R. V. Moody, and B. Solomyak, *Pure point dynamical and diffraction spectra*, Ann. Henri Poincaré **3** (2002), no. 5, 1003–1018. (page 5)

[LS05] E. Lindenstrauss and K. Schmidt, *Symbolic representations of nonexpansive group automorphisms*, Israel J. Math. **149** (2005), 227–266, Probability in mathematics. (page 6)

[LW88] D. A. Lind and T. Ward, *Automorphisms of solenoids and p-adic entropy*, Ergodic Theory Dynam. Systems **8** (1988), no. 3, 411–419. (page 9)

[LW03] J. C. Lagarias and Y. Wang, *Substitution Delone sets*, Discrete Comput. Geom. **29** (2003), no. 2, 175–209. (page 83)

[Min14] M. Minervino, *Dynamics of reducible Pisot substitutions*, preprint, 2014. (page 11, 14, 77)

[Moo97] R.V. Moody, *Meyer sets and their duals*, The mathematics of long-range aperiodic order (Waterloo, ON, 1995), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 489, Kluwer Acad. Publ., Dordrecht, 1997, pp. 403–441. (page 28, 39)

[Mos92] B. Mossé, *Puissances de mots et reconnaissabilité des points fixes d'une substitution*, Theoret. Comput. Sci. **99** (1992), no. 2, 327–334. (page 19)

[MS14] M. Minervino and W. Steiner, *Tilings for Pisot beta numeration*, Indagationes Mathematicae (2014), to appear, `arXiv:1310.1277`. (page 11, 13, 14, 53)

[MT14] M. Minervino and J. M. Thuswaldner, *The geometry of non-unit Pisot substitutions*, Ann. Inst. Fourier (Grenoble) **64** (2014), to appear, `arXiv:1402.2002`. (page 11, 12, 33, 55)

[MW88] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 811–829. (page 40)

[Neu99] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999. (page 25, 30)

[Par60] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416. (page 23)

[Pra99] B. Praggastis, *Numeration systems and Markov partitions from self-similar tilings*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3315–3349. (page 6, 13)

[Que10] M. Queffélec, *Substitution dynamical systems—spectral analysis*, second ed., Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 2010. (page 2, 18, 94)

[Rau82] G. Rauzy, *Nombres algébriques et substitutions*, Bull. Soc. Math. France **110** (1982), no. 2, 147–178. (page 3)

[Rau84] ———, *Ensembles à restes bornés*, Seminar on number theory, 1983–1984 (Talence, 1983/1984), Univ. Bordeaux I, Talence, 1984, pp. Exp. No. 24, 12. (page 97)

[Rau88] ———, *Rotations sur les groupes, nombres algébriques, et substitutions*, Séminaire de Théorie des Nombres (Talence, 1987-1988), Exp. No. 21, Univ. Bordeaux I, Talence, 1988. (page 8)

[Rev91] J.-P. Reveillès, *Géométrie discrète, calcul en nombres entiers et algorithmique*, Thèse de Doctorat, Université Louis Pasteur, Strasbourg, 1991. (page 4)

[Rob00] A. M. Robert, *A course in p-adic analysis*, Graduate Texts in Mathematics, vol. 198, Springer-Verlag, New York, 2000. (page 32)

[RV99] D. Ramakrishnan and R. J. Valenza, *Fourier analysis on number fields*, Graduate Texts in Mathematics, vol. 186, Springer-Verlag, New York, 1999. (page 26)

[RZ10] L. Ribes and P. Zalesskii, *Profinite groups*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 40, Springer-Verlag, Berlin, 2010. (page 36)

[Sad08] L. Sadun, *Topology of tiling spaces*, University Lecture Series, vol. 46, American Mathematical Society, Providence, RI, 2008. (page 2)

[SAI01] Y. Sano, P. Arnoux, and S. Ito, *Higher dimensional extensions of substitutions and their dual maps*, J. Anal. Math. **83** (2001), 183–206. (page 14, 77, 78)

[Sch80] K. Schmidt, *On periodic expansions of Pisot numbers and Salem numbers*, Bull. London Math. Soc. **12** (1980), no. 4, 269–278. (page 24, 54, 75)

[Sch00] ———, *Algebraic coding of expansive group automorphisms and two-sided beta-shifts*, Monatsh. Math. **129** (2000), no. 1, 37–61. (page 6)

[Sen95] M. Senechal, *Quasicrystals and geometry*, Cambridge University Press, Cambridge, 1995. (page 5)

[Ser79] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979, Translated from the French by Marvin Jay Greenberg. (page 26)

[Sid03] N. Sidorov, *Arithmetic dynamics*, Topics in dynamics and ergodic theory, London Math. Soc. Lecture Note Ser., vol. 310, Cambridge Univ. Press, Cambridge, 2003, pp. 145–189. (page 66)

[Sie03] A. Siegel, *Représentation des systèmes dynamiques substitutifs non unimodulaires*, Ergodic Theory Dynam. Systems **23** (2003), no. 4, 1247–1273. (page 9, 43)

[Sin68] J. G. Sinaĭ, *Markov partitions and U-diffeomorphisms*, Funkcional. Anal. i Priložen **2** (1968), no. 1, 64–89. (page 5)

[Sin06a] B. Sing, *Iterated function systems in mixed Euclidean and* $\mathfrak{p}$*-adic spaces*, Complexus mundi, World Sci. Publ., Hackensack, NJ, 2006, pp. 267–276. (page 9)

[Sin06b] ———, *Pisot substitutions and beyond*, Ph.D. thesis, University of Bielefeld, 2006, http://nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:hbz:361-11555. (page 9, 12, 35, 39, 42, 45, 70)

[Sma67] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. (page 9)

[Sol97] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 695–738. (page 2)

[Sol07] ———, *Eigenfunctions for substitution tiling systems*, Probability and number theory—Kanazawa 2005, Adv. Stud. Pure Math., vol. 49, Math. Soc. Japan, Tokyo, 2007, pp. 433–454. (page 2)

[SS02] V. F. Sirvent and B. Solomyak, *Pure discrete spectrum for one-dimensional substitution systems of Pisot type*, Canad. Math. Bull. **45** (2002), no. 4, 697–710, Dedicated to Robert V. Moody. (page 7)

[ST09] A. Siegel and J. M. Thuswaldner, *Topological properties of Rauzy fractals*, Mém. Soc. Math. Fr. (N.S.) (2009), no. 118, 140. (page 4, 10, 13, 25, 46, 47, 53)

[ST13] W. Steiner and J. M. Thuswaldner, *Rational self-affine tiles*, Trans. Amer. Math. Soc. (2013), to appear. (page 13, 53, 62, 63)

[SW02] V. F. Sirvent and Y. Wang, *Self-affine tiling via substitution dynamical systems and Rauzy fractals*, Pacific J. Math. **206** (2002), no. 2, 465–485. (page 4)

[Thu89] W. Thurston, *Groups, tilings and finite state automata*, AMS Colloquium lectures (1989). (page 2, 3, 69)

[Thu06] J. M. Thuswaldner, *Unimodular Pisot substitutions and their associated tiles*, J. Théor. Nombres Bordeaux **18** (2006), no. 2, 487–536. (page 7)

[VS08] A. M. Vershik and B. Solomyak, *The adic realization of the Morse transformation and the extension of its action to the solenoid*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **360** (2008), no. Teoriya Predstavlenii, Dinamicheskie Sitemy, Kombinatornye Metody. XVI, 70–90, 296. (page 9)

[Wal82] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York, 1982. (page 2, 18)

[Wei95] A. Weil, *Basic number theory*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the second (1973) edition. (page 26, 27)

[Wil74] R. F. Williams, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 169–203. (page 9)

Index