MAXIMAL INEQUALITIES FOR STOCHASTIC CONVOLUTIONS DRIVEN BY COMPENSATED POISSON RANDOM MEASURES IN BANACH SPACES

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Abstract. Let $(E, \|\cdot\|)$ be a Banach space such that, for some $q \geq 2$, the function $x \mapsto \|x\|^q$ is of C^2 class and its first and second Fréchet derivatives are bounded by some constant multiples of $(q-1)$-th power of the norm and $(q-2)$-th power of the norm and let S be a C_0-semigroup of contraction type on $(E, \|\cdot\|)$. We consider the following stochastic convolution process

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z), \quad t \geq 0,$$

where $\tilde{N}$ is a compensated Poisson random measure on a measurable space $(Z, \mathcal{Z})$ and $\xi : [0,\infty) \times \Omega \times Z \to E$ is an $\mathbb{F} \otimes \mathcal{Z}$-predictable function. We prove that there exists a càdlàg modification a $\tilde{u}$ of the process u which satisfies the following maximal inequality

$$\mathbb{E} \sup_{0 \leq s \leq t} \|\tilde{u}(s)\|^{q'} \leq C\, \mathbb{E} \left(\int_0^t \int_Z \|\xi(s,z)\|^p \, N(\mathrm{d}s,\mathrm{d}z) \right)^{\frac{q'}{p}},$$

for all $q' \geq q$ and $1 < p \leq 2$ with $C = C(q,p)$.

1. Introduction

Maximal inequalities for stochastic convolutions in the setting of Hilbert spaces or finite dimensional spaces have received considerable attention for many years. Ichikawa [16] considered maximal inequalities for C_0-semigroups of contractions and right continuous martingales in Hilbert spaces, see also Tubaro [35]. A submartingale type inequality for stochastic convolutions of C_0-semigroups of contractions and square integrable martingales, also in Hilbert spaces, was obtained by Kotelenez [21]. Kotelenez proved the existence of a càdlàg version of the stochastic convolution process for square integrable càdlàg martingales. In a paper by the second named author and Peszat [5], the authors established a maximal inequality in a certain class of Banach spaces for the stochastic convolution process driven by a Wiener process. Recently, this maximal inequality was generalized by van Neerven and the first named author to C_0-contraction semigroups on 2-smooth Banach spaces. Since many results obtained in the Wiener case may fail in pure jump type models, maximal inequalities for compensated Poisson random measures deserve an independent investigation.

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Here we extend the results from [5] to the case where the stochastic convolution is driven by a compensated Poisson random measure. We work in the framework of Itô stochastic integrals and convolutions driven by compensated Poisson random measures recently introduced by the second and third authors in [3].

Let us now briefly present the content of the paper. In the first section (i.e. section 2) we set up notations and terminologies and then summarize without proofs some of the standard facts on Itô stochastic integrals with values in martingale type p, $p \in (1, 2]$, Banach spaces, driven by a compensated Poisson random measure $\tilde{N}$ on a measurable space $(Z, \mathcal{Z})$. Section 3 is devoted to the study of the stochastic convolution process $(u(t))_{t\geq 0}$ driven by $\tilde{N}$ defined by the following formula

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\, \tilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geq 0, \tag{1.1}$$

where $S(t)$, $t \geq 0$ is a C_0-contraction semigroup (with the infinitesimal generator A) on a martingale type p, $p \in (1, 2]$, Banach space E and $\xi : [0, \infty) \times \Omega \times Z \to E$ is an $\mathbb{F} \otimes \mathcal{Z}$-predictable function such that for all $T > 0$, $\int_0^T \int_Z \mathbb{E}|f(t,z)|_E^p\, \nu(\mathrm{d}z)\, \mathrm{d}t < \infty$. In particular, we show that there exists a predictable version of the stochastic convolution process u. Under some suitable assumptions we show that the process u is a unique strong solution to the following stochastic evolution equation

$$\begin{aligned} du(t) &= Au(t)\, \mathrm{d}t + \int_Z \xi(t,z)\, \tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \geq 0, \\ u(0) &= 0. \end{aligned} \tag{1.2}$$

In section 4 we present our main results. In particular, the maximal inequalities are stated and proved when the q-th power, for some $q \geq p$, of some equivalent norm on E is of C^2 class and its first and second Fréchet derivatives are bounded by some constant multiples of $(q-1)$-th power of the norm and $(q-2)$-th power of the norm and $S(t)$, $t \geq 0$ is a C_0-contraction semigroup on E with respect to this equivalent norm. For the readers convenience let us state this result.

Theorem 1.1. *Suppose that E is a real separable Banach space satisfying Assumption 4.1 and $S(t)$, $t \geq 0$ is a C_0 semigroup on E satisfying Assumption 4.2. In the above described framework, there exists a separable and càdlàg modification $\tilde{u}$ of the process u defined by formula* (4.1)*. Moreover, for every $q' \geq q$, where p and q are the numbers from Assumption 4.1, there exists a constant C independent of the process ξ, such that for every stopping time $\tau > 0$ and every $t > 0$,*

$$\mathbb{E} \sup_{0 \leq s \leq t\wedge\tau} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^{t\wedge\tau} \int_Z |\xi(s,z)|_E^p\, N(\mathrm{d}s, \mathrm{d}z) \right)^{\frac{q'}{p}}, \ t \geq 0. \tag{1.3}$$

Note that under above assumption on E, it can be shown, see Appendix A, that the Banach space E satisfying the above condition is of martingale type p, for all $p \in (1, 2]$. Hence both the Itô stochastic integral and the stochastic convolution process are well defined.

Let us point out an important consequence of the above result. Namely, if the right-hand side above is finite, then the stochastic convolution process u admits an càdlàg modification, see [7] and a positive result given in [22] for somehow related results in the Hilbert space framework.

In the last part of section 4 we formulate and prove a different version of the maximal inequality. To be more precise, if $p \in [\sqrt{2}, 2]$ and $n \geq [\frac{\ln q}{\ln p}]$, then for every stopping time $\tau > 0$,

$$\mathbb{E} \sup_{0 \leq s \leq t\wedge\tau} |\tilde{u}(s)|_E^{p^n} \leq C \sum_{k=1}^{n} \mathbb{E} \left(\int_0^{t\wedge\tau} \int_Z |\xi(s,z)|_E^{p^k} \,\nu(\mathrm{d}z)\mathrm{d}s \right)^{p^{n-k}}, \; t \geq 0. \tag{1.4}$$

In a brief section 5, we present extensions of the previous result to progressively measurable integrands.

Remark 1.2. *It is possible to prove inequality* (1.1) *by the method based on the Szeköfalvi-Nagy Theorem on unitary dilations, used earlier in* [14]*, see inequality (4) therein. The latter result has recently been generalized to Banach spaces of finite cotype by Fröhlich and Weis* [12]*. However, this method works only for analytic semigroups of contraction type. The results from the current paper are valid for all C_0-semigroups of contraction type. To be more precise, if E is a Banach space of martingale type p , with $1 < p \leq 2$, and A generates an analytic semigroup of contraction type on E, then, by following almost the same lines as in* [14]*, one can prove that for every $T > 0$ there exists a constant $C > 0$ such that for all progressively measurable processes ξ*

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C \; \mathbb{E} \left(\int_0^{t} \int_Z |\xi(s,z)|_E^{p} \, N(\mathrm{d}s, \mathrm{d}z) \right)^{\frac{q'}{p}}, \; t \in [0,T].$$

Let us finish this Introduction by commenting that the results presented are applicable to non-linear SPDEs, e.g. stochastic Euler Equations. In the case of similar problems with the Gaussian noise, the paper [5] on which to a large extent our current research is based on, was in some sense a byproduct of a previous study by the same authors for stochastic Euler Equations in [6]. It turns out that applications to stochastic Navier-Stokes Equations of our paper even before it's publication have been found in a recent paper by Fernando et al. [11]. For related results for stochastic reaction diffusion equations obtained by different approach one can consult a paper [23] by Marinelli and Röckner.

2. Stochastic integral

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, be a filtered probability space satisfying the usual hypothesis. Let $(S, \mathcal{S})$ be a measurable space. We write $\mathbb{N}$ for the set of all natural numbers and set $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We denote by $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ the space of all $\bar{\mathbb{N}}$-valued measures on $(S, \mathcal{S})$ and $\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(S))$ the smallest σ-field on $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ with respect to which all the mapping $i_B : \mathbb{M}_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S}$, are measurable.

Definition 2.1. *A Poisson random measure on $(S, \mathcal{S})$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a map $N : \Omega \to \mathbb{M}_{\bar{\mathbb{N}}}(S)$ such that the family $\{N(B) : B \in \mathcal{S}\}$ of $\mathcal{F}$-measurable $\bar{\mathbb{N}}$-valued functions defined by $N(B) := i_B \circ N : \Omega \to \bar{\mathbb{N}}$ satisfies the following conditions*

(1) *for any* $B \in \mathcal{S}$ *such that* $\eta(B) := \mathbb{E}(N(B)) < \infty$, $N(B)$ *is a Poisson random variable with parameter* $\eta(B)$, *i.e.*

$$\mathbb{P}(N(B) = n) = e^{-\eta(B)} \frac{\eta(B)^n}{n!}, \qquad n = 0, 1, 2, \cdots ;$$

(2) *(independently scattered property) for any pairwise disjoint sets* $B_1, \cdots, B_n \in \mathcal{S}$ *such that* $\eta(B_i) < \infty$, $i = 1, \cdots, n$, *the random variables*

$$N(B_1), \ \cdots, \ N(B_n)$$

are independent.

Remark 2.2. *In what follows we will often assume that* $T \in (0, \infty) \cup \{\infty\}$. *Then, if* $T = \infty$, *by* $[0, T]$ *we mean the half-line* $[0, \infty)$ *and* $\mathcal{F}_T$ *stands for* $\mathcal{F}$. *Similarly, for* $a < \infty$, $(a, \infty]$ *(respectively* $[a, \infty]$*) stands for* (a, ∞) *(respectively* $[a, \infty)$*).*

Definition 2.3. *Let us fix* $T \in (0, \infty) \cup \{\infty\}$. *Let* $\mathcal{P}$ *denote the* σ*-field on* $[0, T] \times \Omega$ *generated by all left-continuous and* $\mathbb{F}$*-adapted real valued processes. We call* $\mathcal{P}$ *the predictable* σ*-field.*
Assume that $(Z, \mathcal{Z})$ *is a measurable space. Let* $\hat{\mathcal{P}}$ *denote the* σ*-field on* $[0, T] \times \Omega \times Z$ *generated by all functions* $g : [0, T] \times \Omega \times Z \to \mathbb{R}$ *satisfying the following properties*

(1) *for every* $t \in [0, T]$, *the mapping* $(\omega, z) \mapsto g(t, \omega, z)$ *is* $\mathcal{F}_t \otimes \mathcal{Z}/\mathcal{B}(\mathbb{R})$*-measurable,*
(2) *for every* (ω, z), *the path* $t \mapsto g(t, \omega, z)$ *is left-continuous.*

We say that an E*-valued process* $g : [0, T] \times \Omega \to E$ *is predictable if it is* $\mathcal{P}/\mathcal{B}(E)$*-measurable.*
We say that a function $f : [0, T] \times \Omega \times Z \to E$ *is* $\mathbb{F} \otimes \mathcal{Z}$*-predictable if it is* $\hat{\mathcal{P}}/\mathcal{B}(E)$*-measurable.*

Remark 2.4. *The predictable* σ*-field* $\mathcal{P}$ *is also generated by the family* $\mathcal{R}$ *(see for instance Th. 3.3 in* [24]*) defined by*

$$\mathcal{R} = \{\{0\} \times F : F \in \mathcal{F}_0\} \cup \{(s, t] \times F : F \in \mathcal{F}_s, 0 \leq s < t, t \in [0, T]\}.$$

The sets belonging to the family $\mathcal{R}$ *are usually called predictable rectangles. Similarly, one can show, see* [39], *that the* $\mathbb{F} \otimes \mathcal{Z}$*-predictable* σ*-field* $\hat{P}$ *is generated by a family* $\hat{R}$

$$\hat{\mathcal{R}} = \{\{0\} \times F \times B : F \in \mathcal{F}_0, B \in \mathcal{Z}\} \cup \{(s, t] \times F \times B : F \in \mathcal{F}_s, B \in \mathcal{Z}, 0 \leq s < t \leq T\}.$$

Note that a function $f : [0, T] \times \Omega \times Z \to E$ *which is now called* $\mathbb{F} \otimes \mathcal{Z}$*-predictable, in* [39] *was called* $\mathbb{F}$*-predictable. We believe that our current terminology is more natural.*

Suppose that $(Z, \mathcal{Z})$ is a measurable space and ν is a non-negative σ-finite measure on it. Let Leb be the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. According to [33], there exists a Poisson random measure N on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$ with the parameter $\eta(B) = \mathbb{E}N(B) = \mathrm{Leb} \otimes \nu(B)$, for $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z}$. In particular, $\eta(I \times A) = \mathrm{Leb}(I)\nu(A)$, for $I \in \mathcal{B}(\mathbb{R}_+)$ and $A \in \mathcal{Z}$. Here as usual we shall employ the notation

$$\tilde{N} = N - \mathrm{Leb} \otimes \nu$$

to denote the compensated Poisson random measure of N.

For $T \in (0,\infty)\cup\{\infty\}$, let $\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$ denote the linear space consisting of (equivalence classes of) all $\mathbb{F}\otimes\mathcal{Z}$-predictable functions $f:[0,T]\times\Omega\times Z\to E$ such that

$$\int_0^T\int_Z \mathbb{E}|f(t,z)|_E^p\,\nu(\mathrm{d}z)\,\mathrm{d}t<\infty. \tag{2.1}$$

In other words, $\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$ is the usual L^p space of E-valued functions on $[0,T]\times\Omega\times Z$ with respect to the σ-field $\hat{\mathcal{P}}$ and the measure $\mathrm{Leb}\otimes\mathbb{P}\otimes\nu$. By $\mathcal{M}^p_{\mathrm{loc}}([0,\infty)\times Z;\hat{\mathcal{P}};E)$ we denote a linear space consisting of all $\mathbb{F}\otimes\mathcal{Z}$-predictable functions $f:[0,\infty)\times\Omega\times Z\to E$ such that condition (2.1) is satisfied for all $T>0$.

Till the end of this section, we will briefly sketch how one constructs, the integral

$$\int_0^T\int_Z f(t,z)\,\tilde{N}(\mathrm{d}t,\mathrm{d}z),\ \text{ for every function } f\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E).$$

This integral we shall call the Itô integral with respect to the compensated Poisson random measure $\tilde{N}$. Full details of the definition can be found in [39].

Definition 2.5. *A function $f:[0,T]\times\Omega\times Z\to E$ is called a step function if there exists a finite sequence of numbers $0=t_0<t_1<\cdots<t_n=T$ and a finite family A^k_{j-1}, $j=1,\cdots,n$, $k=1,\cdots,m$, of sets from $\mathcal{Z}$ with $\nu(A^k_{j-1})<\infty$ such that*

$$f(t,\omega,z)=\sum_{k=1}^m\sum_{j=1}^n\xi^k_{j-1}(\omega)1_{(t_{j-1},t_j]}(t)1_{A^k_{j-1}}(z),\quad (t,\omega,z)\in[0,T]\times\Omega\times Z, \tag{2.2}$$

where ξ^k_{j-1} is an E-valued $\mathcal{F}_{t_{j-1}}$-measurable random variable, for every $j=1,\cdots,n$ and $k=1,\cdots,m$, and for each $j=1,\cdots,n$, the sets A^k_{j-1}, $k=1,\cdots,m$, are pairwise disjoint.

Note that each step function as defined in Definition 2.5 is $\hat{\mathcal{P}}/\mathcal{B}(E)$-measurable. In other words, every step function is $\mathbb{F}\otimes\mathcal{Z}$-predictable. The class of all step functions satisfying (2.1) will be denoted by $\mathcal{M}^p_{step}([0,T]\times Z;\hat{\mathcal{P}};E)$.

Definition 2.6. *Assume that f is a step function in $\mathcal{M}^p_{step}([0,T]\times Z;\hat{\mathcal{P}};E)$ of the form (2.2). If $t\in[0,T]$, then the Itô integral over the interval $[0,t]$ of a step function f in $\mathcal{M}^p_{step}([0,T]\times Z;\hat{\mathcal{P}};E)$ of the form (2.2) with respect to $\tilde{N}$ is a random variable $I_t(f)$, defined by*

$$I_T(f):=\sum_{k=1}^m\sum_{j=1}^n\xi^k_{j-1}(\omega)\tilde{N}((t_{j-1}\wedge t,t_j\wedge t]\times A^k_{j-1}).$$

Note that, for every $f\in\mathcal{M}^p_{step}([0,T]\times Z;\hat{\mathcal{P}};E)$, $I_T(f)$ is linear with respect to f and satisfies the following inequality, see Lemma C.2 in [3] and, for related results, a recent paper [10] by Dirksen,

$$\mathbb{E}\,|I_T(f)|_E^p\le C\,\mathbb{E}\int_0^t\int_Z|f(s,z)|_E^p\,\nu(\mathrm{d}z)\,\mathrm{d}s, \tag{2.3}$$

where C is the same constant as the one in the martingale-type-p property of the space E. Moreover, the process $I_t(f)$, $t\in[0,T]$ is an E-valued, mean 0 and càdlàg $\mathbb{F}$-martingale.

The definition of the Itô integral can be in a straightforward way extended to the set $\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$. As usual, for f in that class, the value of this extension will be denoted by $I_T(f)$ and called the Itô integral of f with respect to the compensated Poisson random measure $\tilde{N}$. Moreover, for $f \in \mathcal{M}^p([0,T] \times Z; \hat{\mathcal{P}}; E)$, the Itô integral from 0 to t is defined by

$$I_t(f) = \int_0^t \int_Z f(s,z)\, \tilde{N}(\mathrm{d}s, \mathrm{d}z) = I_T(1_{(0,t]}f), \quad t \in [0,T].$$

Let $f \in \mathcal{M}^p([0,T] \times Z; \hat{\mathcal{P}}; E)$. It was shown in [3] and [39] (see also [32] for the case $p = 2$) that the process $I_t(f)$, $t \in [0,T]$ is a càdlàg p-integrable $\mathbb{F}$-martingale with mean 0. In particular, $I_t(f)$ has a modification which has $\mathbb{P}$-a.s. càdlàg trajectories and satisfies the following inequality

$$\mathbb{E}|I_t(f)|_E^p = \mathbb{E}\Big| \int_0^t \int_Z f(s,z)\, \tilde{N}(\mathrm{d}s, \mathrm{d}z)\Big|_E^p \leq C\, \mathbb{E} \int_0^t \int_Z |f(s,z)|_E^p\, \nu(\mathrm{d}z)\, \mathrm{d}s. \tag{2.4}$$

From now on, while considering the stochastic process $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$, $t \in [0,T]$, we will assume that it has $\mathbb{P}$-a.s. càdlàg trajectories.

Remark 2.7. *We have defined the Itô integral for integrands belonging to the class $\mathcal{M}^p([0,T] \times Z; \hat{\mathcal{P}}; E)$ of predictable processes. One should note that in the paper* [3] *by the second and third authors, the integral is defined for an analogous class of progressively measurable processes. See also* [32]. *This approach is also discussed in section 5 of the present paper.*

If τ is a stopping time with $\mathbb{P}\{\tau \leq T\} = 1$, we may set, for $\omega \in \Omega$,

$$I_\tau(f)(\omega) = I_t(f)(\omega) \ \text{ with } t = \tau(\omega). \tag{2.5}$$

Analogously, we shall also use the notation $I_\tau(f) =: \int_0^\tau \int_Z f(s,z)\tilde{N}(ds,dz)$. In this case, one can show that

$$\int_0^\tau \int_Z f(s,z)\, \tilde{N}(\mathrm{d}s, \mathrm{d}z) = \int_0^T \int_Z 1_{(0,\tau]}(s) f(s,z)\, \tilde{N}(\mathrm{d}s, \mathrm{d}z), \ \mathbb{P}\text{-a.s.}. \tag{2.6}$$

Remark 2.8. *Note that since τ is a stopping time (without any additional property), the random process*

$$1_{(0,\tau]} : [0,\infty) \times \omega \ni (s,\omega) \mapsto 1_{(0,\tau(\omega]}(s) \in [0,1]$$

is predictable, see the comment after [30, Definition IV.5.3] *or Proposition 4.6 in* [24]. *Hence, provided that the process f belongs to $\mathcal{M}^p([0,T] \times Z; \hat{\mathcal{P}}; E)$, the process $1_{(0,\tau]}f$ belongs to that space as well. In particular, the integral on the RHS of* (2.6) *is well defined.*

Let us observe that for every stopping time τ with $\mathbb{P}(\tau < \infty) = 1$, we have

$$\int_0^{t\wedge\tau} \int_Z f(s,z)\, \tilde{N}(\mathrm{d}s, \mathrm{d}z) = \int_0^t \int_Z 1_{(0,\tau]}(s)\, f(s,z)\, \tilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \in [0,T]. \tag{2.7}$$

We will also need the following result about the integral $\int_0^t \int_Z f(s,\omega,z) N(\mathrm{d}s, \mathrm{d}z)(\omega)$, which is defined, for every $\omega \in \Omega$, as Bochner integral with respect to measure $N(\mathrm{d}s, \mathrm{d}z)(\omega)$ on $[0,t] \times Z$.

For the notion and properties of a point processes let us refer the reader to [17], [4] or [39]. Let us assume that π is a stationary Poisson point process on $(Z, \mathcal{Z})$ with the intensity measure ν, see [31, Theorem 54] for the existence of such a process. For simplicity of notation, the Poisson random measure associated to the Poisson point process π will be still denoted by N. We use the notation $\tilde{N}(t,A) = N(t,A) - t\nu(A)$, $t \geq 0$, $A \in \mathcal{Z}$ to denote its compensated Poisson random measure.

Proposition 2.9. *If $T \in (0,\infty)\cup\{\infty\}$ and $f : [0,T]\times\Omega\times Z \to E$ is a $\mathcal{B}([0,T])\otimes\mathcal{F}_T\otimes\mathcal{Z}$-measurable function and*

$$\mathbb{E}\int_0^T\int_Z |f(s,z)|_E\, N(\mathrm{d}s,\mathrm{d}z) < \infty, \tag{2.8}$$

then we have for every $t \in [0,T]$,

$$\int_0^t\int_Z f(s,\omega,z)N(\mathrm{d}s,\mathrm{d}z)(\omega) = \sum_{s\le t} f(s,\omega,\pi(s,\omega)),\ \mathbb{P}-a.s. \tag{2.9}$$

Proof. Note that for every $\mathcal{B}([0,T])\otimes\mathcal{F}_T\otimes\mathcal{Z}$-measurable function satisfying (2.8), one can always find a set Ω_0 with $\mathbb{P}(\Omega_0) = 1$ and a sequence $\{f^n\}$ of simple functions on $[0,T]\times Z$ of the form $\sum_{i=1}^m x_i 1_{B_i}$, $x_i \in E$ and $B_i \in \mathcal{B}([0,T])\otimes\mathcal{Z}$ such that $|f^n(t,\omega,z) - f(t,\omega,z)|_E$ decreases to 0 as $n\to\infty$, for all $(t,z)\in[0,T]\times Z$, for all $\omega\in\Omega_0$. It can be easily verified that (2.9) holds true for every simple function of the form $\sum_{i=1}^m a_i 1_{B_i}$. □

3. Stochastic convolution

In this section, we continue to assume that E is a separable Banach space of martingale type p, where $p \in (1,2]$. Let $(S(t))_{t\ge0}$ be a contraction C_0-semigroup on E with the infinitesimal generator A. Let us denote by $R(\lambda,A) = (\lambda I - A)^{-1}$, $\lambda > 0$, the resolvent operator of A and by $A_\lambda = \lambda A(\lambda I - A)^{-1}$ the Yosida approximation of A. It is well known that A_λ is a bounded operator on E and $A_\lambda x \to x$, as $\lambda\to\infty$, for $x\in\mathcal{A}$ and $\lambda R(\lambda,A)x \to x$, as $\lambda\to\infty$, for all $x\in E$. Moreover, $\lambda R(\lambda,A)x \in \mathcal{D}(A)$, for all $x \in E$.

Suppose that $\xi \in \mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$ and $\tilde{N}$ is a compensated Poisson random measure corresponding to the point process $\pi = (\pi(t))_{t\ge0}$. The aim of this paper is to study the path properties of the stochastic convolution process u defined by

$$u(t) = \int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z),\quad t\in[0,T]. \tag{3.1}$$

Now let us consider Problem (1.2), which for the convenience of the reader we rewrite below.

$$\begin{aligned} \mathrm{d}u(t) &= Au(t)\,\mathrm{d}t + \int_Z \xi(t,z)\,\tilde{N}(\mathrm{d}t,\mathrm{d}z),\quad t\in[0,T],\\ u(0) &= 0. \end{aligned} \tag{3.2}$$

In this chapter we will study the above Question under a stronger assumption on the process ξ, i.e. that $\xi \in \mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};\mathcal{D}(A))$.

Definition 3.1. *Suppose that $\xi \in \mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};\mathcal{D}(A))$. A strong solution to Problem* (3.2) *on the time interval $[0,T]$ is a $\mathcal{D}(A)$-valued $\mathbb{F}$-adapted stochastic process $(u(t))_{t\in[0,T]}$ with E-valued càdlàg trajectories such that*

(1) *$u(0) = 0$ a.s.*
(2) *For any $t\in[0,T]$ the following equality holds $\mathbb{P}$-a.s.*

$$u(t) = \int_0^t Au(s)\,\mathrm{d}s + \int_0^t\int_Z \xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z). \tag{3.3}$$

Similarly we can define a strong solution to Problem (3.2) *if* $T=\infty$ *and* $\xi \in \mathcal{M}^p_{\rm loc}([0,\infty)\times Z;\hat{\mathcal{P}};\mathcal{D}(A))$.

Lemma 3.2. *Assume that* $\xi\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};\mathcal{D}(A))$ *for some* $T>0$. *Then the process* u *defined by*

$$u(t)=\int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t\geq 0, \tag{3.4}$$

is a unique strong solution of Equation (1.2). *In particular,* u *has* E*-valued càdlàg trajectories.*

Proof. Let us fix $T>0$ and $t\in[0,T]$. Define a function $F:[0,t]\times E\ni(s,x)\mapsto S(t-s)x\in E$. By the continuity of F and the $\mathbb{F}\otimes\mathcal{Z}$-predictability assumption on ξ, we infer that the composition mapping

$$[0,t]\times\Omega\times Z\ni(s,\omega,z)\mapsto(s,\xi(s,\omega,z))\mapsto F(s,\xi(s,\omega,z))\in E$$

is $\mathbb{F}\otimes\mathcal{Z}$-predictable. So the process $S(t-s)\xi(s,z)$ belongs to $\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$. Hence, the process defined by

$$\int_0^r\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z),\quad r\in[0,t],$$

is a $\mathbb{F}$-martingale on $[0,t]$, see [39]. Therefore, $u(t)$ is $\mathcal{F}_t$-measurable. Since $\xi\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};\mathcal{D}(A))$ and $R(\lambda,A)A=\lambda R(\lambda,A)-I_E$ on $\mathcal{D}(A)$, we infer

$$\begin{aligned}R(\lambda,A)\int_0^t\int_Z AS(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)&=\lambda R(\lambda,A)\int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\&\quad-\int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

It follows that $u(t)\in\mathcal{D}(A)$.

Next we shall show that

$$A\int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)=\int_0^t\int_Z AS(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z),\quad \mathbb{P}\text{-a.s..} \tag{3.5}$$

Let us take $h\in(0,t)$. By applying (2.4), we find

$$\begin{aligned}&\mathbb{E}\left|A\int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)-\int_0^t\int_Z AS(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|_E^p\\&\leq 2^p\,\mathbb{E}\left|A\int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)-\frac{S(h)-I}{h}\int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|_E^p\\&\quad+2^pC_p\,\mathbb{E}\int_0^t\int_Z\left|AS(t-s)\xi(s,z)-\frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p\nu(\mathrm{d}z)\,\mathrm{d}s\\&=:\mathrm{I}(h)+\mathrm{II}(h).\end{aligned}\tag{3.6}$$

Clearly, the integrand $\left|AS(t-s)\xi(s,z)-\frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p$ of $\mathrm{II}(h)$ is bounded by a function $C|A\xi(s,z)|_E^p$, with some constant C, and it converges to 0 pointwise on $[0,t]\times\Omega\times Z$. It follows from the Lebesgue Dominated Convergence Theorem (Lebesgue DCT for short) that $\mathrm{II}(h)$ converges to 0 as $h \searrow 0$. Hence (3.5) holds. Therefore, applying Fubini's Theorem, see [39], gives

$$\begin{aligned}\int_0^t Au(s)\,\mathrm{d}s &= \int_0^t \int_0^s \int_Z AS(s-r)\xi(r,z)\tilde{N}(\mathrm{d}r,\mathrm{d}z)\,\mathrm{d}s\\ &= \int_0^t \int_Z \int_r^t AS(s-r)\xi(r,z)\,\mathrm{d}s\,\tilde{N}(\mathrm{d}r,\mathrm{d}z)\\ &= u(t) - \int_0^t \int_Z \xi(r,z)\,\tilde{N}(\mathrm{d}r,\mathrm{d}z), \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

from which we can also see that u is a càdlàg process. The uniqueness of the solution follows from a standard argument, see arXiv:1005.1600v4 or [39] for more details. □

4. Maximal inequalities for stochastic convolution

From now on we make the following assumptions on the Banach space E.

Assumption 4.1. *Suppose that $\big(E,|\cdot|\big)$ is a real separable Banach space and $p \in (1,2]$. In addition we assume that the Banach space E satisfies the following condition:*
There exists a norm $|\cdot|_E$ on E which is equivalent to $|\cdot|$, and numbers $q \in [p,\infty)$ such that the function $\phi : E \ni x \mapsto |x|_E^q \in \mathbb{R}$, is of class C^2 and there exist constants k_1, k_2 such that for every $x \in E$, the first the second Fréchet derivatives of ϕ satisfy, respectively, $|\phi'(x)| \le k_1|x|_E^{q-1}$ and $|\phi''(x)| \le k_2|x|_E^{q-2}$, for all $x \in E$.

Assumption 4.2. *The C_0 semigroup $S(t)$, $t \ge 0$ on E is of contraction type with respect to the norm $|\cdot|_E$ from Assumption 4.1.*

Now we proceed with the study of the stochastic convolution

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t \in [0,T], \tag{4.1}$$

provided $T > 0$ and the process ξ belongs to the space $\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$.

Let us now formulate our main result a proof of which will be presented at the end of this section and preceded by two Lemmata: 4.7 and 4.8.

Theorem 4.3. *Suppose that $\big(E,|\cdot|\big)$ is a real separable Banach space satisfying Assumption 4.1, with numbers $p \in (1,2]$ and $q \in [p,\infty)$ and $S(t)$, $t \ge 0$ is a C_0 semigroup on E satisfying Assumption 4.2. Assume that $\xi \in \mathcal{M}^p_{\mathrm{loc}}([0,\infty)\times Z;\hat{\mathcal{P}};E)$ and in fact that*

$$\mathbb{E}\left(\int_0^T \int_Z |\xi(s,z)|^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q}{p}},\ \ T > 0.$$

Then there exists a separable and càdlàg modification $\tilde{u}$ of the process u defined by formula (4.1). Moreover, for every $q' \ge q$, there exists a constant C independent of the process ξ, such that for every stopping time $\tau > 0$ and every $t > 0$,

$$\mathbb{E}\sup_{0\le s\le t\wedge\tau} |\tilde{u}(s)|^{q'} \le C\,\mathbb{E}\left(\int_0^{t\wedge\tau} \int_Z |\xi(s,z)|^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q'}{p}}. \tag{4.2}$$

Before proceeding with proofs, let us point our an important ingredient of the above result, i.e. the càdlg̀ property of the stochastic convolution process u. This topic has attracted recently an attention even in the Hilbert space setup because of the counterexample presented in [7] and a positive result given in [22].

Remark 4.4. *It is worth pointing out that since by* [9, Proposition 3.6]*, every adapted and stochastically continuous process on an interval* $[0,T]$ *has a predictable version on* $[0,T]$*, we conclude that the process* $u(t)$*,* $t\geq 0$ *has a predictable version. Henceforth, when we study the stochastic convolution process, we refer to the version of it that is càdlàg and its supremum over every compact interval* $[0,T]$ *is* $\mathcal{F}_T$*-measurable.*

Remark 4.5. *It can be proved, see Appendix A, that if the real separable Banach space* E *satisfies Assumption* (4.1)*, then* E *is of martingale type* p*, for all* $p\in(1,2]$*. Hence the stochastic convolution process* (4.1) *is well defined for* $\xi\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$ *and in particular, if* $\xi\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};\mathcal{D}(A))$*, then* (4.1) *is a unique strong solution of equation* (1.2) *by Lemma 3.2.*

Remark 4.6. *Note that if* $\mathcal{O}$ *is a domain in* $\mathbb{R}^n$ *with Lipschitz boundary* $\partial\mathcal{O}$*, then the Sobolev spaces* $H^{s,r}(\mathcal{O})$ *with* $r\in[2,\infty)$ *and* $s\in\mathbb{R}_+$ *satisfy Assumption* (4.1) *and the Lebesgue* $L^r(\mathcal{O})$*-spaces with* $r\geq 2$ *also satisfies Assumption* (4.1)*.*

Before proving the main theorem, we first need the following Lemmas.

Lemma 4.7. *If Assumptions 4.1 and 4.2 are satisfied, then for all* $x\in D(A)$*,*

$$\phi'(x)(Ax)\leq 0.$$

Proof. This follows immediately from the fact that the function $t\mapsto\phi(S(t)x)$ is decreasing and

$$\left.\frac{d\phi(S(t)x)}{dt}\right|_{t=0}=\phi'(S(0)x)(Ax)=\phi'(x)(Ax).$$

□

In the remainder of this section we will always assume that Assumptions 4.1 and 4.2 are satisfied.

Lemma 4.8. *Suppose that* $\xi\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$ *for some* $T>0$*. Then there exists a version* $\bar{u}$ *of the process* u *defined by equality* (4.1) *such that the function* $\sup_{t\in[0,T]}|\bar{u}(t)|$ *is* $\mathcal{F}_T$*-measurable.*

Proof. According to Remark 4.5, we can take $p\in(1,2]$. Let us fix $T>0$ and that $\xi\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$. By applying the inequality (2.4) we have, for $0\leq r<t\leq T$,

$$\begin{aligned}\mathbb{E}|u(t)-u(r)|_E^p&\leq 2^p\,\mathbb{E}\left|\int_r^t\int_Z S(t-s)\xi(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|_E^p\\&\quad+2^p\,\mathbb{E}\left|\int_0^r\int_Z\Big(S(t-s)-S(r-s)\Big)\xi(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|_E^p\\&\leq 2^pC_p\,\mathbb{E}\int_0^T\int_Z 1_{(r,t]}(s)|\xi(s,z)|_E^p\,\nu(\mathrm{d}z)\,\mathrm{d}s\\&\quad+2^pC_p\,\mathbb{E}\int_0^T\int_Z|1_{(0,r]}\big(S(t-s)-S(r-s)\big)\xi(s,z)|_E^p\,\nu(\mathrm{d}z)\,\mathrm{d}s.\end{aligned}\tag{4.3}$$

Clearly, both terms on the right hand side of above inequality converge to 0 as $t \searrow r$ or $r \nearrow t$. Therefore, we conclude that the process u is stochastically continuous. Hence, as the space E is separable, according to Theorem 5.3 in [38], we can find a version $\bar{u}$ of u which is separable. That is there exists a countable subset T_0 which is dense in $[0,T]$ such that $\bar{u}(t)$ belongs to the set of partial limits $\{\lim_{s\in T_0, s\to t} \bar{u}(s)\}$, for all $t \in [0,T]\backslash T_0$. Hence

$$\sup_{t\geq 0} |\bar{u}(t)| = \sup_{t\in[0,T]} \lim_{s_n\to t, s_n\in T_0} |\bar{u}(s_n)| = \sup_{s_n\in T_0} |\bar{u}(s_n)|.$$

Since $\sup_{s_n\in T_0} |\bar{u}(s_n)|$ is $\mathcal{F}_T$-measurable, we deduce that the function $\sup_{t\geq 0} |\bar{u}(t)|$ is also $\mathcal{F}_T$-measurable. □

We now ready to embark on the proof of the main result.

Proof of Theorem 4.3. Let us first notice that because the original norm $|\cdot|$ is equivalent to the norm $|\cdot|_E$, it is sufficient to prove inequality (4.2) with the latter norm, i.e.

$$\mathbb{E} \sup_{0\leq s\leq t\wedge\tau} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^{t\wedge\tau} \int_Z |\xi(s,z)|_E^p \, N(\mathrm{d}s, \mathrm{d}z) \right)^{\frac{q'}{p}}. \tag{4.4}$$

Secondly, let us note that it is sufficient to consider processes defined on a bounded time intervals. Hence we fix $T > 0$ and $\xi \in \mathcal{M}^p([0,T]\times Z; \hat{\mathcal{P}}; E)$. In view of Remark 4.5, let us also fix $p \in (1,2]$ and let us fix $q \in [p,\infty)$ as in Assumption 4.1.

Case I. We first prove (4.2) for $\xi \in \mathcal{M}^p([0,T]\times Z; \hat{\mathcal{P}}; \mathcal{D}(A))$. We have shown in Lemma 3.2 that the process u is a unique strong solution to Problem (3.2) satisfying

$$u(t) = \int_0^t Au(s)\,\mathrm{d}s + \int_0^t \int_Z \xi(s,z) \tilde{N}(\mathrm{d}s,\mathrm{d}z), \ t \in [0,T]. \tag{4.5}$$

Since the function $\phi : E \ni x \mapsto |x|_E^q$ is of C^2 class by assumption, one may apply the Itô formula from [15], see also [39, Theorem 3.5.3], to the process u given by (4.5) and get, for $t \geq 0$,

$$\begin{aligned}\phi(u(t)) &= \int_0^t \phi'(u(s))(Au(s))\,\mathrm{d}s + \int_0^t \int_Z \phi'(u(s-))(\xi(s,z))\, \tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &\quad + \int_0^t \int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s,z))\Big] N(\mathrm{d}s,\mathrm{d}z) \ \ \mathbb{P}\text{-a.s..}\end{aligned} \tag{4.6}$$

Let $\tau \geq 0$ be a stopping time. Since by Lemma 4.7, $\phi'(x)(Ax) \leq 0$, for all $x \in D(A)$, we infer that

$$\begin{aligned}\phi(u(t\wedge\tau)) &\leq \int_0^t \int_Z 1_{(0,\tau]}(s)\phi'(u(s-))(\xi(s,z))\, \tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &\quad + \int_0^{t\wedge\tau} \int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s,z))\Big] N(\mathrm{d}s,\mathrm{d}z) \\ &=: I_1(t) + I_2(t) \ \ \mathbb{P}\text{-a.s.}, \ \ t\geq 0.\end{aligned} \tag{4.7}$$

Note that $I_1(t)$ is an $\mathbb{R}$-valued local martingale. Applying the real-valued version of Burkholder-Davis-Gundy inequality, see [20, Proposition 15.7], to the process I_1 we deduce for some constant

C that

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}|I_1(t)|_E &\le C\,\mathbb{E}\left(\int_0^T\int_Z 1_{(0,\tau]}(s)|\phi'(u(s-))(\xi(s,z))|_E^2\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{1}{2}}\\
&= C\,\mathbb{E}\Big(\sum_{t\le T}1_{(0,\tau]}(s)|\phi'(u(s-))(\xi(s,\pi(s)))|_E^2\Big)^{\frac{1}{2}}\\
&\le C\,\mathbb{E}\Big(\sum_{t\le T}1_{(0,\tau]}(s)|\phi'(u(s-))(\xi(s,\pi(s)))|_E^p\Big)^{\frac{1}{p}} \qquad (4.8)\\
&= C\,\mathbb{E}\left(\int_0^T\int_Z 1_{(0,\tau]}(s)|\phi'(u(s-))(\xi(s,z))|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{1}{p}}\\
&\le k_1C\,\mathbb{E}\sup_{0\le t\le T\wedge\tau}|u(t)|_E^{q-1}\left(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Next, by applying Hölder's and Young's inequalities to the process I_1, we infer

$$
\begin{aligned}
\mathbb{E}\sup_{t\ge0}|I_1(t)|_E &\le k_1C\left(\mathbb{E}\Big[\sup_{0\le t\le T\wedge\tau}|u(t)|_E^{q-1}\Big]^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\left(\mathbb{E}\left(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&\le k_1C\,\frac{q-1}{q}\,\varepsilon\,\mathbb{E}\sup_{0\le t\le T\wedge\tau}|u(t)|_E^q+k_1C\frac{1}{\varepsilon^{q-1}q}\mathbb{E}\left(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q}{p}}. \qquad (4.9)
\end{aligned}
$$

To estimate the integral $I_2(t)$, we observe first that for every $t\ge0$,

$$
\begin{aligned}
|I_2(t)|_E &\le \int_0^{t\wedge\tau}\int_Z\Big|\phi(u(s-)+\xi(s,z))-\phi(u(s-))-\phi'(u(s-))(\xi(s,z))\Big|_E\,N(\mathrm{d}s,\mathrm{d}z)\\
&= \sum_{s\in(0,t\wedge\tau]\cap\mathcal{D}(\pi)}\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))-\phi'(u(s-))(\xi(s,\pi(s)))\Big|_E \quad \mathbb{P}\text{-a.s..}
\end{aligned}
$$

Since the function ϕ is of C^2 class by the assumption, applying the mean value Theorem, see [8], to the function ϕ, for each $s\in[0,t\wedge\tau]$ we have

$$
\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))\Big|_E\le\int_0^1|\xi(s,\pi(s))|_E\Big|\phi'(u(s-)+\theta\xi(s,\pi(s)))\Big|_{\mathcal{L}(E)}\,d\theta.
$$

Since $|x+\theta y|_E\le\max\{|x|_E,|x+y|_E\}$ for all $x,y\in E$ and $\theta\in[0,1]$, and, by Assumption 4.1, $|\phi'(x)|\le k_1|x|_E^{q-1}$, $x\in E$, we infer

$$
\begin{aligned}
\Big|\phi'(u(s-)+\theta\xi(s,\pi(s)))\Big|_{\mathcal{L}(E)} &\le k_1\big|u(s-)+\theta\xi(s,\pi(s))\big|_E^{q-1}\\
&\le k_1\max\big\{|u(s-)|_E^{q-1},\big|u(s-)+\xi(s,\pi(s))\big|_E^{q-1}\big\}.
\end{aligned}
$$

Observe that for all $0\le s\le t\wedge\tau$,

$$
|u(s-)|_E^{q-1}\le\sup_{0\le r\le t\wedge\tau}|u(r-)|_E^{q-1}\le\sup_{0\le t\le T\wedge\tau}|u(t)|_E^{q-1}.
$$

Moreover, since $u(s-)+\xi(s,\pi(s))=u(s)$, for $s\in(0,t\wedge\tau]\cap\mathcal{D}(\pi)$, we have

$$
|u(s-)+\xi(s,\pi(s))|_E^{q-1}\le\sup_{0\le r\le t\wedge\tau}|u(r)|_E^{q-1}\le\sup_{0\le t\le T\wedge\tau}|u(t)|_E^{q-1}.
$$

Therefore, we deduce that for each $s \in [0, t \wedge \tau]$,

$$\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))\Big|_E \leq k_1|\xi(s,\pi(s))|_E \sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-1}.$$

It follows that

$$\begin{aligned}
&\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))-\phi'(u(s-))(\xi(s,\pi(s)))\Big|_E \\
&\qquad\qquad \leq \Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))\Big|_E+\Big|\phi'(u(s-))(\xi(s,\pi(s)))\Big|_E \\
&\qquad\qquad \leq 2k_1|\xi(s,\pi(s))|_E \sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-1}.
\end{aligned}$$

On the other hand, we can also find some $0<\delta<1$ such that

$$\begin{aligned}
&\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))-\phi'(u(s-))(\xi(s,\pi(s)))\Big|_E \\
&\leq \frac{1}{2}|\xi(s,\pi(s))|_E^2|\phi''(u(s-)+\delta\xi(s,\pi(s)))| \leq \frac{k_2}{2}|\xi(s,\pi(s))|_E^2|u(s-)+\delta\xi(s,\pi(s))|_E^{q-2}.
\end{aligned}$$

Hence, a similar argument as above, we obtain for $s \in [0, t\wedge\tau]$,

$$\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))-\phi'(u(s-))(\xi(s,\pi(s)))\Big|_E \leq \frac{k_2}{2}|\xi(s,\pi(s))|_E^2 \sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-2}.$$

Thus, with $K=(2k_1)^{2-p}\big(\frac{k_2}{2}\big)^{p-1}$, we have

$$\begin{aligned}
&\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))-\phi'(u(s-))(\xi(s,\pi(s)))\Big|_E \\
&\leq \left(2k_1|\xi(s,\pi(s))|_E \sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-1}\right)^{2-p}\left(\frac{k_2}{2}|\xi(s,\pi(s))|_E^2 \sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-2}\right)^{p-1} \\
&\leq K|\xi(s,\pi(s))|_E^p \sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-p}.
\end{aligned}$$

Hence, by Proposition 2.9, we get

$$\begin{aligned}
&\sum_{s\in(0,t\wedge\tau]\cap\mathcal{D}(\pi)}\Big|\phi(u(s-)+\xi(s,\pi(s)))-\phi(u(s-))-\phi'(u(s-))(\xi(s,\pi(s)))\Big|_E \\
&\leq K \sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-p}\int_0^{T\wedge\tau}\int_Z|\xi(r,z)|_E^p\,N(\mathrm{d}r,\mathrm{d}z).
\end{aligned}$$

Therefore, collecting all these together yields

$$\begin{aligned}
\mathbb{E}\sup_{t\geq 0}|I_2(t)|_E &\leq \int_0^{T\wedge\tau}\int_Z\Big|\phi(u(r-)+\xi(r,z))-\phi(u(r-))-\phi'(u(r-))(\xi(r,z))\Big|_E N(\mathrm{d}r,\mathrm{d}z) \\
&\leq K\sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^{q-p}\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z) \\
&\leq K\left(\mathbb{E}\sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^q\right)^{\frac{q-p}{q}}\left(\mathbb{E}\left(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q}{p}}\right)^{\frac{p}{q}} \\
&\leq K\frac{q-p}{q}\varepsilon\,\mathbb{E}\sup_{0\leq t\leq T\wedge\tau}|u(t)|_E^q+K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\,\mathbb{E}\left(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^q,
\end{aligned} \tag{4.10}$$

where we used again Hölder's inequality and Young's inequality and the constant K depending only on k_1, k_2, p and q . Combining (4.9) and (4.10), we obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T\wedge\tau}|u(t)|_E^q \;\le\; & \left(k_1C\frac{q-1}{q}+K\frac{q-p}{q}\right)\varepsilon\,\mathbb{E}\sup_{0\le t\le T\wedge\tau}|u(t)|_E^q \\ & +\Big(k_1C\frac{1}{\varepsilon^{q-1}q}+K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\Big)\,\mathbb{E}\Big(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\Big)^{\frac{q}{p}}.\end{aligned}$$

Now we can choose a suitable positive number ε such that

$$\left(k_1C\frac{q-1}{q}+K\frac{q-p}{q}\right)\varepsilon=\frac{1}{2}.$$

Consequently, there exists C which is independent of A such that

$$\mathbb{E}\sup_{0\le t\le T\wedge\tau}|u(s)|_E^q\le C\mathbb{E}\left(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q}{p}}. \tag{4.11}$$

Case II. Now suppose $\xi\in\mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$. Set $R(n,A)=(nI-A)^{-1}$, $n\in\mathbb{N}$. Then we put $\xi^n(t,\omega,z)=nR(n,A)\xi(t,\omega,z)$ on $[0,T]\times\Omega\times Z$. By the Hille-Yosida Theorem, $\|R(n,A)\|\le\frac{1}{n}$ and $\xi^n(t,\omega,z)\in\mathcal{D}(A)$, for every $(t,\omega,z)\in[0,T]\times\Omega\times Z$. Moreover, $\xi^n(t,\omega,z)\to\xi(t,\omega,z)$ pointwise on $[0,T]\times\Omega\times Z$. Also, we observe that $|\xi^n-\xi|=|nR(n,A)\xi-\xi|\le 2|\xi|$. Therefore, by applying the Lebesgue DCT, it follows that $\mathbb{P}$-a.s.

$$\mathbb{E}\int_0^T\int_Z 1_{(0,\tau]}(s)\,|\xi^n(s,z)-\xi(s,z)|_E^p\,\nu(\mathrm{d}z)\mathrm{d}s\;\to\;0\text{ as }n\to\infty.$$

Since the Poisson random measure N is a $\mathbb{P}$-a.s. positive and

$$\mathbb{E}\int_0^T\int_Z 1_{(0,\tau]}(s)\,|\xi^n(s,z)-\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)=\mathbb{E}\int_0^T\int_Z 1_{(0,\tau]}|\xi^n(s,z)-\xi(s,z)|_E^p\,\nu(\mathrm{d}z)\mathrm{d}s,$$

we see that $\mathbb{P}$-a.s.

$$\int_0^T\int_Z 1_{(0,\tau]}(s)\,|\xi^n(s,z)-\xi(s,z)|^p\,N(\mathrm{d}s,\mathrm{d}z)\to 0,\quad\text{as }n\to\infty. \tag{4.12}$$

Let us fix $n\in\mathbb{N}$. Clearly, $\xi^n\in\mathcal{M}^p([0,T]\times Z;\mathcal{D}(A))$ and so we may define a process u^n by

$$u^n(t)=\int_0^t S(t-s)\xi^n(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t\in[0,T].$$

Since by Lemma 3.2, the process u^n is a strong solution of equation (1.2) with the process ξ replaced by ξ^n, we infer that it is an E-valued càdlàg. According to inequality (4.11), for every stopping time $\tau\ge 0$ the following inequality holds

$$\mathbb{E}\sup_{0\le t\le T\wedge\tau}|u^n(t)|^q\le C\,\mathbb{E}\left(\int_0^{T\wedge\tau}\int_Z|\xi^n(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q}{p}}.$$

On the other hand, since by inequality (2.4), we have

$$\begin{aligned}\mathbb{E}|u^n(t)-u(t)|_E^p&=\mathbb{E}\left|\int_0^t\int_Z\Big(S(t-s)\xi^n(s,z)-S(t-s)\xi(s,z)\Big)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|_E^p\\ &\le C_p\,\mathbb{E}\int_0^T\int_Z|\xi^n(s,z)-\xi(s,z)|_E^p\,\nu(\mathrm{d}z)\,\mathrm{d}s,\ t\in[0,T],\end{aligned} \tag{4.13}$$

we deduce that for $t \in [0,T]$, $u^n(t)$ converges to $u(t)$ in $L^p(\Omega)$. Moreover, by (4.11), we have

$$\mathbb{E}\sup_{t\geq 0}|u^n(t)-u^m(t)|_E^q \leq C\mathbb{E}\left(\int_0^T\int_Z|\xi^n(s,z)-\xi^m(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q}{p}}. \tag{4.14}$$

An argument similar to the one used in (4.12) shows that the right hand-side of (4.14) converges to 0 as $n,m\to\infty$. Hence, on the basis of Chebyshev inequality and the Borel-Cantelli Lemma, it is possible to choose a sequence $\{n_k\}_{k=1}^\infty$ of natural numbers such that the series of càdlàg processes

$$\sum_{k=1}^\infty [u^{n_{k+1}}(t)-u^{n_k}(t)],\ \in [0,T]$$

converges, $\mathbb{P}$-a.s., uniformly on $[0,T]$, to a càdlàg process which we shall denote by $\tilde{u}=(\tilde{u}(t))_{t\geq 0}$. In view of Lemma 4.8, we may assume that the process $\tilde{u}$ is separable. Thus, the function $\sup_{t\geq 0}|\tilde{u}(t)|^q$ is also measurable. Moreover, we have

$$\mathbb{E}\sup_{t\geq 0}|u^{n_k}(t)-\tilde{u}(t)|_E^q \to 0, \quad \text{as } n_k\to\infty. \tag{4.15}$$

Therefore, by the Minkowski Inequality and inequality (4.11), we have

$$\begin{aligned}\Big[\mathbb{E}\sup_{0\leq s\leq T\wedge\tau}|\tilde{u}(t)|_E^q\Big]^{\frac{1}{q}} &\leq \Big[\mathbb{E}\sup_{0\leq t\leq T\wedge\tau}|\tilde{u}(t)-u^{n_k}(t)|_E^q\Big]^{\frac{1}{q}} + \Big[\mathbb{E}\sup_{0\leq t\leq T\wedge\tau}|u^{n_k}(t)|_E^q\Big]^{\frac{1}{q}}\\ &\leq \Big[\mathbb{E}\sup_{0\leq t\leq T\wedge\tau}|\tilde{u}(t)-u^{n_k}(t)|_E^q\Big]^{\frac{1}{q}} + \Big[C\,\mathbb{E}\Big(\int_0^{T\wedge\tau}\int_Z|\xi^{n_k}(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}},\end{aligned}$$

where the constant C does not depend on the operator A and so remains the same for every n. It follows by letting $n_k\to\infty$ in above inequality that

$$\mathbb{E}\sup_{0\leq t\leq T\wedge\tau}|\tilde{u}(t)|^q \leq C\,\mathbb{E}\Big(\int_0^{T\wedge\tau}\int_Z|\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\Big)^{\frac{q}{p}}.$$

Also, by Minkowski inequality and Hölder's inequality we have for every $t\geq 0$,

$$\begin{aligned}\big(\mathbb{E}|\tilde{u}(t)-u(t)|_E^p\big)^{\frac{1}{p}} &\leq \big(\mathbb{E}|\tilde{u}(t)-u^{n_k}(t)|_E^p\big)^{\frac{1}{p}} + \big(\mathbb{E}|u(t)-u^{n_k}(t)|_E^p\big)^{\frac{1}{p}}\\ &\leq \big(\mathbb{E}|\tilde{u}(t)-u^{n_k}(t)|_E^q\big)^{\frac{1}{q}} + \big(\mathbb{E}|u(t)-u^{n_k}(t)|_E^p\big)^{\frac{1}{p}}\\ &\leq \left(\mathbb{E}\sup_{0\leq t\leq T}|\tilde{u}(t)-u^{n_k}(t)|_E^q\right)^{\frac{1}{q}} + \big(\mathbb{E}|u(t)-u^{n_k}(t)|_E^p\big)^{\frac{1}{p}}.\end{aligned}$$

Letting $n\to\infty$, it follows from (4.13) and (4.15) that $u(t)=\tilde{u}(t)$ in $L^p(\Omega)$ for any $t\geq 0$. This shows the inequailty (4.2) for $q'=q$. The case $q'>q$ follows from the fact that if Banach space E satisfies Assumption 4.1 for some q, then Condition 1 is also satisfied with $q'>q$.

□

The following result could be derived immediately from the proof of the above result.

Corollary 4.1. *Let E be a Banach space satisfying Assumption 4.1. Then the stochastic convolution process u has a càdlàg modification.*

Corollary 4.2. *Let E be a Banach space satisfying Assumption 4.1. Let $\sqrt{2} \le p \le 2$. Then for any $n \in \mathbb{N}$ with $p^n \ge q$, there exists a constant $C = C(E,n)$ such that for every $\xi \in \bigcap_{k=1}^n \mathcal{M}^{p^k}_{\rm loc}([0,\infty)\times Z;\hat{\mathcal{P}};E)$ and for every stopping time $\tau > 0$ and $t \ge 0$,*

$$\mathbb{E} \sup_{0\le s\le t\wedge\tau} |\tilde{u}(s)|_E^{p^n} \le C \sum_{k=1}^n \mathbb{E}\left(\int_0^{t\wedge\tau}\int_Z |\xi(s,z)|_E^{p^k}\,\nu(\mathrm{d}z)\mathrm{d}s\right)^{p^{n-k}}, \tag{4.16}$$

where $\tilde{u}$ is a separable and càdlàg modification of u as before.

Two essential ingredients of that proof are formulated below.

Lemma 4.9. *Let E be a martingale type p Banach space, $1 < p \le 2$, satisfying Assumption 4.1. Let $\tau > 0$ be a stopping time. For any $q' \ge q$, there exists a constant C such that, for all $\xi \in \mathcal{M}^p_{\rm loc}([0,\infty)\times Z;\hat{\mathcal{P}};E)$, we have*

$$\mathbb{E} \sup_{0\le s\le t\wedge\tau} \left|\int_0^s\int_Z \xi(r,z)\,\tilde{N}(\mathrm{d}r,\mathrm{d}z)\right|_E^{q'} \le C\,\mathbb{E}\left(\int_0^{t\wedge\tau}\int_Z |\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{q'}{p}},\ t\ge 0. \tag{4.17}$$

This result is a special case of Theorem 4.3 with $S(t) = I$, $t \ge 0$.

Lemma 4.10. *Let $\sqrt{2} \le p \le 2$. For any $n \in \mathbb{N}$ there exists a constant $D_n > 0$ such that for any process*

$$f \in \bigcap_{k=1}^n \mathcal{M}^{p^k}_{\rm loc}([0,\infty)\times Z;\hat{\mathcal{P}};\mathbb{R}),$$

and all $t \ge 0$ and stopping times $\tau > 0$, the following inequality holds

$$\mathbb{E} \sup_{0\le s\le t\wedge\tau} \left|\int_0^s\int_Z f(r,z)\,\tilde{N}(\mathrm{d}r,\mathrm{d}z)\right|^{p^n} \le D_n \sum_{k=1}^n \mathbb{E}\left(\int_0^{t\wedge\tau}\int_Z |f(s,z)|^{p^k}\,\nu(\mathrm{d}z)\mathrm{d}s\right)^{p^{n-k}}. \tag{4.18}$$

The proof of Lemma 4.10 is similar to the proof [2, Lemma 5.2] or [36, Lemma 4.1] and hence it will be omitted.

Proof of Corollary 4.2. Let us take $n \in \mathbb{N}$. By applying first Theorem 4.3 and next Lemma 4.10 when $\xi \in \mathcal{M}^p([0,T]\times Z;\hat{\mathcal{P}};E)$, we deduce that for all $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}|\tilde{u}(s)|_E^{p^n} &\le C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p\,N(\mathrm{d}s,\mathrm{d}z)\right)^{p^{n-1}}\\
&\le 2^{p^{n-1}}C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right)^{p^{n-1}}\\
&\quad +2^{p^{n-1}}C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p\,\nu(\mathrm{d}z)\,\mathrm{d}s\right)^{p^{n-1}}\\
&\le C(n)\sum_{k=1}^n \mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^{p^k}\,\nu(\mathrm{d}z)\,\mathrm{d}s\right)^{p^{n-k}},
\end{aligned}$$

Here, we used in the third inequality Lemma 4.10 with f replaced by real-valued process ξ such that

$$|\xi|_E^p \in \bigcap_{k=1}^{n-1} \mathcal{M}_{\mathrm{loc}}^{p^k}([0,\infty) \times Z; \hat{\mathcal{P}}; \mathbb{R}).$$

This completes the proof of Corollary 4.2. □

5. Extension to progressively measurable integrands

Corollary 4.2 can be generalized to integrands which are progressively measurable processes. Let us recall that a process $\xi : [0,T]\times\Omega\times Z \to E$ is $\mathbb{F}\otimes\mathcal{Z}$-progressively measurable, if ξ is $\mathcal{BF}\otimes\mathcal{Z}/\mathcal{B}(E)$-measurable, where, see [38, section 6.5], $\mathcal{BF}$ is the σ-field consisting of all sets $A \subset [0,T] \times \Omega$ such that for every $t \in [0,T]$, the set $A \cap \big([0,t] \times \Omega\big)$ belongs to the sigma field $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$. Note that $\mathcal{BF} \otimes \mathcal{Z}$ is the σ-field generated by a family of all sets $A \subset [0,T] \times \Omega \times Z$ such that for every $t \in [0,T]$, the set $A \cap \big([0,t] \times \Omega \times Z\big)$ belongs to the sigma field $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t \otimes \mathcal{Z}$.
For $p \in [1,\infty)$, the set of all of p–integrable $\mathcal{BF} \otimes \mathcal{Z}$-progressively processes $\xi : [0,T] \times \Omega \times Z \to E$ will be denoted by

$$\mathcal{M}^p([0,T] \times Z; \mathcal{BF} \otimes \mathcal{Z}; E)$$

and the Banach space of all equivalence classes of p–integrable $\mathcal{BF} \otimes \mathcal{Z}$-progressively processes $\xi : [0,T] \times \Omega \times Z \to E$ will be denoted by

$$\mathbb{M}^p([0,T] \times Z; \mathcal{BF} \otimes \mathcal{Z}; E).$$

As noted in Remark 2.7, the Itô integral with respect to a compensated Poisson random measure of processes from the class has been introduced in [3], see also [39, Theorem 3.2.27]. The following follows from [39, Theorem 3.2.27].

Proposition 5.1. *If $p \in [1,\infty)$ and a progressively measurable process $\xi : [0,T] \times \Omega \times Z \to E$ belongs to $\mathcal{M}^p([0,T] \times Z; \mathcal{BF} \otimes \mathcal{Z}; E)$ then there exists a sequence of càglàd step functions $\xi_n \in \mathcal{M}^p_{step}([0,T] \times Z; \hat{\mathcal{P}}; E)$, such that $\xi_n \to \xi$ in $\xi \in \mathcal{M}^p([0,T] \times Z; \mathcal{BF} \otimes \mathcal{Z}; E)$, as $n \to \infty$.*

Corollary 5.1. *Let E be a Banach space satisfying Assumption 4.1. Let $\sqrt{2} \le p \le 2$ and $n \in \mathbb{N}$ such that $p^n \ge q$. Then for for every $\xi \in \cap_{k=1}^n \mathcal{M}^{p^k}([0,T] \times Z; \mathcal{BF} \otimes \mathcal{Z}; E)$ there exists a process $\tilde{u}$ which is a separable and càdlàg modification of the stochastic convolution process u defined, as before, by (3.1). Moreover, there exists a constant $C = C(E,n)$ independent of ξ, such that for every stopping time τ and $t \in [0,T]$, inequality (4.16) holds true.*

Proof. By Proposition 5.1, there exists a sequence $\{\xi_i : i \in \mathbb{N}\} \subset \cap_{k=1}^n \mathcal{M}^{p^k}_{step}([0,T] \times Z; \hat{\mathcal{P}}; E)$ of càglàd processes convergent to ξ in $\cap_{k=1}^n \mathcal{M}^{p^k}([0,T] \times Z; \mathcal{BF} \otimes \mathcal{Z}; E)$. By Theorem 4.3, for every i, the exists a separable càdlàg modification $\tilde{u}_i$ of the process u_i being the solution of the Problem

$$u_i(t) = \int_0^t \int_Z S(t-s)\xi_i(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t \in [0,T]. \tag{5.1}$$

which satisfies

$$\mathbb{E} \sup_{0\le s\le t\wedge\tau} |\tilde{u}_i(s)|_E^{p^n} \le C \sum_{k=1}^{n} \mathbb{E}\left(\int_0^{t\wedge\tau} \int_Z |\xi_i(s,z)|_E^{p^k}\, \nu(\mathrm{d}z)\mathrm{d}s \right)^{p^{n-k}}, \ t\in[0,T], \quad l\in\mathbb{N}. \tag{5.2}$$

and, for all $i,j\in\mathbb{N}$,

$$\mathbb{E} \sup_{0\le s\le t\wedge\tau} |\tilde{u}_i(s)-\tilde{u}_j|_E^{p^n} \le C \sum_{k=1}^{n} \mathbb{E}\left(\int_0^{t\wedge\tau} \int_Z |\xi_i(s,z)-\xi_j(s,z)|_E^{p^k}\, \nu(\mathrm{d}z)\mathrm{d}s \right)^{p^{n-k}}, \ t\in[0,T], \quad l\in\mathbb{N}. \tag{5.3}$$

Arguing as in the proof of Theorem 4.3 we can conclude the proof. □

6. Final comments

Inequality (1.1) can also be derived by the method used by the third named author and Seidler in [14], see as inequality (4) therein. These authors used the Szeköfalvi-Nagy's Theorem on unitary dilations in Hilbert spaces. However, this method works only for analytic semigroups of contraction type while the results from the current paper are valid for all C_0 semigroups of contraction type. Let us now formulate the following result whose proof is a clear combination of the proofs from [14] and [12]. For the explanation of the terms used we refer the reader to the latter work. Similar observation for processes driven by a Wiener process was made independently by Seidler [34].

Theorem 6.1. *Let E be a martingale type p Banach space, where $1<p\le 2$. Let $-A$ be a generator of a bounded analytic semigroup in E such that for some $\theta<\frac{1}{2}\pi$, the operator A has a bounded $H^\infty(S_\theta)$ calculus. Then, for any $0<q'<\infty$, there exists a constant C such that for all $\xi\in\mathcal{M}^p_{\mathrm{loc}}([0,\infty)\times Z;\hat{\mathcal{P}};E)$ and for every stopping time $\tau>0$, we have*

$$\mathbb{E}\sup_{0\le s\le t\wedge\tau} \left| \int_0^s \int_Z S(s-r)\xi(r,z)\,\tilde{N}(\mathrm{d}r,\mathrm{d}z) \right|_E^{q'} \le C\,\mathbb{E}\left(\int_0^{t\wedge\tau}\int_Z |\xi(s,z)|_E^p\, N(\mathrm{d}s,\mathrm{d}z) \right)^{\frac{q'}{p}}, \quad t\ge 0. \tag{6.1}$$

The following result could be derived immediately from the proof of above theorem.

Corollary 6.1. *Let E be a martingale type p Banach space, where $1<p\le 2$. Let $-A$ be a generator of a bounded analytic semigroup in E such that for some $\theta<\frac{1}{2}\pi$ the operator A has a bounded $H^\infty(S_\theta)$ calculus. Then, the stochastic convolution process u defined by (1.1) has càdlàg modification.*

Appendix A. Appendix

Definition A.1. *A Banach space E with norm $\|\cdot\|$ is of martingale type p, for $p\in(1,2]$ if and only if there exists a constant $C_p(E)>0$ such that for any E-valued discrete martingale $\{M_k\}_{k=1}^n$*

the following inequality holds

$$\mathbb{E}\|M_n\|^p \le C_p(E) \sum_{k=0}^{n} \mathbb{E}\|M_k - M_{k-1}\|^p, \tag{A.1}$$

with $M_{-1} = 0$ as usual.

The following definition of 2-smooth Banach spaces in terms of asymptoticity of the modulus of smoothness of the norm can be found in [28] and [29].

Definition A.2. *A Banach space E is p-smooth if there exists an equivalent norm defined by the modulus of smoothness of $(E, \|\cdot\|)$*

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 : \|x\| = \|y\| = 1\}$$

satisfying $\rho_E(t) \le Kt^p$ for all $t > 0$ and some $K > 0$.

Remark A.3. *A Banach space is of martingale type p if and only if it is p-smooth, see* [28]*. Hence all spaces $L^q(\mu)$, for $q \in [p,\infty)$ and $q > 1$ with an arbitrary positive measure μ are of martingale type p. Note that any closed subspaces of martingale type p spaces are of martingale type p. So the Sobolev spaces $W^{k,q}$, for $q \in [p,\infty)$ and $k > 0$ are of martingale type p.*

The following Lemma can be found in [37].

Lemma A.4. *A Banach space E is p-smooth, $1 < p \le 2$, if and only if the Fréchet derivative of the norm function $x \mapsto \|x\|^p$ is globally $(p-1)$-Hölder continuous on E.*

Lemma A.5. *If a real separable Banach space E satisfies Assumption* (4.1)*, then E is of martingale type p, for all $p \in (1,2]$.*

Proof of Lemma A.5. see [29] It is sufficient to consider the case $p = 2$, see [29]. Let E be a Banach space with norm $\|\cdot\|$. We assume that $q > 2$ and that the function

$$\psi : E \ni x \mapsto \|x\|^q \in \mathbb{R}$$

is of C^2-class and satisfies the standard assumptions, i.e.

$$\|\psi'(x)\| \le C_1\|x\|^{q-1}, \;\; \|\psi''(x)\| \le C_2\|x\|^{q-2}, \; x \in E.$$

We consider a function

$$\phi : E \ni x \mapsto \|x\|^q \in \mathbb{R}.$$

We claim that ϕ is of C^1 class and of C^2 class on $E \setminus \{0\}$, and ϕ' is globally Lipschitz continuous on E.

To see this, observe first by chain rule that for any $x \in E \setminus \{0\}$,

$$\phi'(x) = \frac{2}{q}\big[\psi(x)\big]^{\frac{2}{q}-1}\psi'(x).$$

Thus,

$$\|\phi'(x)\| \le C\big(\|x\|^q\big)^{\frac{2}{q}-1}\|x\|^{q-1} = C\|x\|.$$

In particular, $\lim_{x\to 0}\|\phi'(x)\| \to 0$ and thus ϕ is differentiable at 0 and $d_0\phi = 0$.

Applying the chain rule again, we have for $x \in E\setminus\{0\}$

$$\phi''(x) = \frac{2}{q}(\frac{2}{q}-1)\big[\psi(x)\big]^{\frac{2}{q}-2}\psi'(x)\otimes\psi'(x) + \frac{2}{q}\big[\psi(x)\big]^{\frac{2}{q}-1}\psi''(x)$$

As above, using the assumptions of the derivatives of ψ we infer that there exists $C>0$ such that

$$\|\phi''(x)\| \le C, \ \ x \in E\setminus\{0\}.$$

For any $x, y \in E\setminus\{0\}$, by applying the mean value Theorem, see e.g. [8], we have

$$\|\phi'(x) - \phi'(y)\| = \|\phi''(\theta)(x-y)\| \le C\|x-y\|,$$

where the point θ lies on the same line segment between x and y. Hence the first derivative ϕ' is globally Lipschitz continuous. By applying Lemma A.4, we infer that the Banach space E is 2-smooth and hence it is of martingale type 2. □

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